Descriptive set theory

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These (currently unfinished) notes are adapted from lectures by Márton Elekes and exercise sessions by Balázs Bursics at Eötvös Loránd University (Budapest) in spring 2026. I thank Alex Kempf for reexplaining many parts to me.

Some proofs not yet written down are replaced with references to Kechris' book *Classical descriptive set theory*.

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[1]Training data includes lectures, exercise sessions, Kechris' book, Wikipedia, and various other online resources.

1. Introduction

Although this might be historically dubious, descriptive set theory is often described as originating in attempts to prove the continuum hypothesis, before it turned out to be independent. Under the form "every subset $H \subseteq \mathbb{R}$ has countable or continuum cardinality", one could try to prove it by increasing topological complexity of H:

- If H is open, this is trivial.
- For H closed, we will deduce it (in ¶ 11.3) from the Cantor–Bendixson theorem (¶ 10.4).
- If H is a countable union of closed subsets, this follows from the previous point.
- What about countable intersections of open subsets? What about more complex subsets?

We will see that all Borel subsets have countable or continuum cardinality (¶ 20.3), so a counterexample to the continuum hypothesis has to be "very complex" topologically.

For another motivation, during the early development of measure theory, Lebesgue famously "proved" that the projection of a Borel subset of the plane onto the real line is Borel. This is completely false! Are such subsets at least Lebesgue-measurable, then? It turns out that the answer is yes, and a proof uses infinite games, a typical tool of descriptive set theory.

In general, the goal of descriptive set theory is to study subsets of nice topological spaces according to the complexity of their topological descriptions. A spectacular type of application is to prove that two conditions are not equivalent by proving that the subsets A and B of elements satisfying the respective conditions have different complexity levels, for example, A is Borel and B is not. In some cases, this has been used to disprove conjectures to which no explicit counterexample is still known.

Many applications hinge upon the fact that meager subsets are ubiquitous. These are subsets which are "small" in a certain topological sense, somewhat like Lebesgue-null subsets in $\mathbb{R}^n$, but at the same time very different. The Baire category theorem states that in completely metrizable spaces, meager subsets have empty interior. For example, in the space of continuous functions $\mathcal{C}([0,1],\mathbb{R})$, only meager-many functions are differentiable somewhere, so there must exist continuous nowhere differentiable functions.

The class of nice topological spaces used in descriptive set theory is Polish spaces, which are defined as separable and completely metrizable topological spaces. Many commonly encountered spaces are Polish, for example: $\mathbb{R}^n$, the closed unit interval $[0,1]$, the space of continuous real-valued functions $\mathcal{C}([0,1])$ (topologized with the infinity norm), the Cantor space $\{0,1\}^{\mathbb{N}}$ (with the product topology), and the Baire space $\mathbb{N}^{\mathbb{N}}$ (idem). A non-example is $\mathbb{Q}$.

Being separable means having a countable dense subset, which is a "smallness" condition, whose necessity is linked to the fact that many of our notions of topological complexity are defined in terms of countability (e.g., Borel subsets are generated from open subsets by complements and countable unions). For example, to prove that the graph of a Borel function is Borel (¶ 23.6), we need to be able to separate points by countably many open subsets in order to express the graph by a Borel condition. It also enables to reduce many problems to standard spaces such as the Baire space $\mathbb{N}^{\mathbb{N}}$.

Being complete means having the topology induced by some complete metric, which is a "regularity" condition, often used to generate points by successive approximations, as in the proof of the Baire category theorem (¶ 15.4). Another example is the fact that every uncountable Polish space embeds the Cantor space $\{0,1\}^{\mathbb{N}}$, proved by essentially mapping each infinite binary sequence to the limit of a Cauchy sequence defined from its finite prefixes (¶ 11.2).

We speak of "completely metrizable spaces" and not "complete metric spaces" because we usually want to view the space not as a metric space, but as a topological space, up to homeomorphism. We merely require that the topology is sufficiently tame by requiring that there exists *some* complete metric which generates it. For example, the open unit interval $(0,1)$ with the standard metric is not complete, yet it is Polish since it is homeomorphic to $\mathbb{R}$.

Moreover, we often go further and view the space not as a topological space but as a measurable space with the Borel σ-algebra. Remarkably, this structure is the same for all Polish spaces of the same cardinality (¶ 25.1, which allows studying the Borel structure of simpler spaces like $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$ using more complicated spaces like $\mathcal{C}([0,1])$. For example, it is not completely trivial to describe a non-Borel subset of $\mathbb{R}$ explicitly (the standard construction, a Vitali set, uses the axiom of choice), but $\mathcal{C}([0,1])$ has a very familiar non-Borel subset, consisting of the differentiable functions (¶ 30.3).

The tools of descriptive set theory include infinite trees, infinite games, and set theory.

2. Terminology and notations

We use the term ***inhabited*** instead of "non-empty" for sets.[2]

In a topological space X, we denote the interior of a subset $H \subseteq X$ by $\operatorname{int}(H)$ and the closure by $\operatorname{cl}(H)$.

In a metric space X, we denote the distance by d and the open ball of radius ε around a point x by $B(x,\varepsilon) := \{y \in X \mid d(x,y) < \varepsilon\}$. A subset of this form is called an ε-ball. Similarly, the closed ball of radius ε around x is $\overline{B}(x,\varepsilon) := \{y \in X \mid d(x,y) \leq \varepsilon\}$ (recall that $\overline{B}(x,\varepsilon)$ is not the closure of $B(x,\varepsilon)$ in general!). We extend the notation to subsets, i.e., $B(H,\varepsilon) := \bigcup_{x\in H} B(x,\varepsilon)$ and $\overline{B}(H,\varepsilon) := \bigcup_{x\in H} B(x,\varepsilon)$. The diameter of a subset $Y \subseteq H$ is $\operatorname{diam}(Y) := \sup_{x,y\in H} d(x,y)$.

We write[3] $X^\star$ for the set of finite sequences of elements of a set X as well as ε for the empty sequence, $u[:n]$ for the finite prefix $(u_0, ..., u_{n-1})$ of an infinite sequence $u \in X^{\mathbb{N}}$, as well as $[s]$ for the set of infinite sequences having the finite sequence s as a prefix, and $s \cdot t$ for concatenation of sequences (where s is finite and t is finite or infinite).

3. Polish spaces

We introduce Polish spaces, the class of nice topological spaces used in descriptive set theory, and recall along the way some useful characterizations of their defining properties.

[2]We work in classical mathematics with choice. Nevertheless, owing to its elegance, we import this term from constructive mathematics, where being inhabited, i.e., having an element, is stronger than being non-empty.

[3]The notation $X^\star$ avoids the conflict with denoting $\mathbb{N} \setminus \{0\}$ as $\mathbb{N}^*$.

(3.1) For a metric space, the following conditions are equivalent:

(1) The space is complete: every Cauchy sequence converges (a sequence is Cauchy when for every $\varepsilon > 0$ there exists an ε-ball containing all terms after a certain index).
(2) Every descending sequence of inhabited closed subsets with diameter tending to 0 has an inhabited intersection (which must then be a singleton).

(3.2) For a metric space, the following conditions are equivalent:

(1) The space is separable: there exists a countable dense subset.
(2) For all $\varepsilon > 0$, there exists a countable ε-net (a cover by ε-balls).
(3) The space is second-countable (it has a countable basis).
(4) The space satisfies the countable chain condition (CCC)[4]: every family of disjoint inhabited open subsets is countable.
(5) The space is Lindelöf (every open cover has a countable subcover).
(6) The space is hereditarily Lindelöf (every subspace is Lindelöf).

In particular, by the characterization via hereditary Lindelöfness, a subspace of a separable metric space is separable.

Proof. We prove (1) $\iff$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (2) and (3) $\implies$ (6) $\implies$ (5) $\implies$ (2). The most interesting implications are:

(4) $\implies$ (2): Take $\varepsilon > 0$ and zornify to obtain a maximal subset C such that the balls $B(c, \varepsilon/2), c \in C$ are all disjoint. Then C is countable by CCC and the balls $B(c, \varepsilon), c \in C$ form an ε-net.

(3) $\implies$ (6): Let (B_n) be a countable basis. Take a subset Y and a family of open subsets $(U_i, i \in I)$ such that $Y \subseteq \bigcup_{i \in I} U_i$. For each n, choose j_n such that $B_n \subseteq U_{j_n}$, if available. The U_{j_n} for n such that j_n is defined form a countable subcover of Y: for every $y \in Y$, we have some i such that $y \in U_i$, and some n such that $y \in B_n \subseteq U_i$, so j_n is defined and $y \in B_n \subseteq U_{j_n}$. □

(3.3) For a metric space, the following conditions are equivalent:

(1) The space is compact: every open cover has a finite subcover.
(2) The space is countably compact: every countable open cover has a finite subcover. Equivalently, every ascending sequence of open subsets whose union is the whole space must contain the whole space already, or dually, every descending sequence of inhabited closed subsets has inhabited intersection.
(3) The space is sequentially compact, i.e., every sequence has a converging subsequence.
(4) The space is complete, and it is totally bounded, i.e., for all $\varepsilon > 0$, there exists a finite ε-net.

Proof. (2) $\implies$ (3): Let (x_n) be a sequence and take a point x in the intersection of the descending sequence of inhabited closed subsets $\mathrm{cl}(\{x_n, n \geq 0\}) \supseteq \mathrm{cl}(\{x_n, n \geq 1\}) \supseteq \cdots$.

(3) $\implies$ (2): Let $F_0 \supseteq F_1 \supseteq \cdots$ be a descending sequence of inhabited closed subsets, take $x_n \in F_n$ and let x be a limit of a subsequence of (x_n). Then $x \in \bigcap_n F_n$.

(3), (2) $\implies$ (1): By total boundedness, we have a countable basis (U_n). Consider an open cover (V_i) and extract a subcover of the countable cover formed by those U_n contained in some V_i.

[4] This should obviously have been called the countable *antichain* condition.

(3) $\iff$ (4): It suffices to prove that a metric space is totally bounded iff every sequence has a Cauchy subsequence.

Assume the space is totally bounded and take a sequence (x_n). Consider a finite 1-net and extract a subsequence whose terms all belong to one same ball of this 1-net, which exists by pigeonhole principle. Let y_0 be the first term of this subsequence, then drop this first term, consider a finite 1/2-net, extract a subsequence again, let y_1 be its first term, etc. Then (y_n) is a Cauchy subsequence of (x_n).

Conversely, let $\varepsilon > 0$ and recursively pick (x_n) such that the ball $B(x_n, \varepsilon/2)$ is disjoint from all the balls $B(x_m, \varepsilon/2), m < n$ while possible. This must terminate, since otherwise no subsequence of (x_n) can be Cauchy, and the $B(x_n, \varepsilon)$ form a finite ε-net. □

(3.4) A ***Polish space*** is a topological space which is completely metrizable (i.e., whose topology can be generated by a complete metric), and separable. In particular, every compact metrizable space is Polish.

(3.5) The product $\prod_n X_n$ of countably many metric spaces $X_n, n \in \mathbb{N}$ can be metrized with $d(x, y) := \sum_n 2^{-n} d(x_n, y_n)$ assuming each X_n is bounded, which we can always ensure without changing the topology by replacing the metric d with $x, y \mapsto \min(1, d(x, y))$. If moreover each X_n is complete then the product $\prod_n X_n$ is complete. If each X_n is separable with $(x_n(i), i \in \mathbb{N})$ dense, then the product is separable: the set of sequences $(x_n(j_n), n \in \mathbb{N})$ with (j_n) ultimately zero is dense. If each X_n is compact then the product is compact, by a diagonal extraction argument: a convergent subsequence of $\left(x_i \in \prod_n X_n, i \in \mathbb{N}\right)$ can be found by extracting a subsequence (x'_i) so that $(x'_i(0))$ converges in X_0, then further extracting from (x'_i) a subsequence (x''_i) so that $(x''_i(1))$ converges in X_1, etc., and then the subsequence $(x_0, x'_1, x''_2, ...)$ of (x_i) converges in $\prod_n X_n$.

In particular, Polish spaces are closed under countable products.

Likewise, the sum (disjoint union) $\sum_n X_n$ of countably many metric spaces $X_n, n \in C$ can be metrized, assuming wlog that each is bounded by 2, by defining the distance between x and y to be their distance in X_n if they belong to the same X_n and 1 otherwise. This preserves completeness and separability, so Polish spaces are closed under countable sums.

(3.6) It follows that the following fundamental spaces are Polish:

- $\mathbb{R}$, $[0, +\infty)$, $[0, 1]$,
- $(0, 1)$ (not complete with its standard metric, but homeomorphic to $\mathbb{R}$),
- $\mathbb{R}^n$ and $[0, 1]^n$ for $n \in \mathbb{N}$,
- $\mathbb{R}^{\mathbb{N}}$,
- The ***Cantor space*** $\{0, 1\}^{\mathbb{N}}$,
- The ***Baire space*** $\mathbb{N}^{\mathbb{N}}$,
- The ***Hilbert cube*** $[0, 1]^{\mathbb{N}}$.

4. Polish subspaces of Polish spaces

We can completely characterize when a subspace of a Polish space is Polish. This is the case for a closed subspace, since the restriction of a complete metric is still complete. However, a non-closed subspace can be Polish as well: consider the subspace $(0, 1)$ of $\mathbb{R}$, which is Polish because it is homeomorphic to $\mathbb{R}$. In general, the Polish subspaces are the so-called G_δ subsets, and the complete metric is found by taking inspiration from this example: we "spread out" the subspace more and more as we get closer to its complement.

(4.1) A ***G_δ subset*** of a topological space is a countable intersection of open subsets, and an ***F_σ subset*** is a countable union of closed subsets.

G_δ subsets are closed under countable intersections and finite unions. Dually, F_σ subsets are closed under countable unions and finite intersections. In a metric space, every closed subset F is G_δ since we can write it as $F = \bigcap_{n\in\mathbb{N}} B(F, 2^{-n})$, and dually, every open subset is F_σ.

(4.2) Let X be Polish and let $Y \subseteq X$ be a G_δ subspace. Then Y is Polish. More specifically, Y embeds as a closed subspace of $X \times \mathbb{R}^{\mathbb{N}}$.

Proof. Write $Y = \bigcap_n U_n$ where U_n is open. Consider the map

$$\begin{aligned} Y &\to X \times \mathbb{R}^{\mathbb{N}} \\ y &\mapsto (y, 1/d(y, U_0^c), 1/d(y, U_1^c), \ldots). \end{aligned}$$

This is an embedding: it is well-defined because the U_n are open (so $d(y, U_n^c) > 0$ for $y \in U_n$), it is continuous, and its inverse is continuous as it is just a projection. That the image is closed means that if $(y_n \in Y)$ is a sequence converging to $y \in X$, and furthermore $(1/d(y_n, U_k^c), n \in \mathbb{N})$ converges for each fixed k, then $y \in Y$. This holds because if $y \notin Y$, say $y \in U_k$, then $1/d(y_n, U_k^c)$ diverges to infinity. $\square$

(4.3) Let $f : X \to Y$ where X is a topological space and Y is a metric space. The ***oscillation*** of f at a point $x \in X$, denoted $\operatorname{osc}_f(x)$, is the infimum of $\operatorname{diam}(f(U))$ over all neighborhoods U of x. We have $\operatorname{osc}_f(x) = 0$ iff f is continuous at x.

(4.4) The continuity points of a function $f : X \to Y$, where X is a topological space and Y a metric space, form a G_δ subset of X.

Proof. This subset is

$$\left\{x \in X \mid \operatorname{osc}_f(x) = 0\right\} = \bigcap_n \left\{x \in X \mid \operatorname{osc}_f(x) < 2^{-n}\right\}$$

and $\left\{x \in X \mid \operatorname{osc}_f(x) < \varepsilon\right\}$ is open for all $\varepsilon > 0$. $\square$

(4.5) Kuratowski's extension theorem: Let X be a metric space, let Y be a complete metric space, let $A \subseteq X$, and let $f : A \to Y$ be continuous. There exists an extension $\tilde{f} : \tilde{A} \to Y$ of f, where the domain $\tilde{A}$ is G_δ and $A \subseteq \tilde{A} \subseteq \operatorname{cl}(A)$.

Proof. The idea is to extend f to each point $x \in \operatorname{cl}(A)$ where a continuous extension is possible. Informally, the neighborhoods of x all contain points from A, whose images are "approximations of possible images of x", and the extension is possible when these approximations define the image of x in a consistent way, which can be expressed using the oscillation, and the new domain will be G_δ for the same reason as in the previous point.

To make this precise, we define the oscillation just as in the total case of total functions, i.e., $\operatorname{osc}_f(x)$ is the infimum of $\operatorname{diam}(f(U))$ over all neighborhoods U of x, and we let $\tilde{A} \coloneqq \left\{x \in \operatorname{cl}(A) \mid \operatorname{osc}_f(x) = 0\right\}$. Note that for $x \in \operatorname{cl}(A) \setminus \tilde{A}$, there is no continuous extension of f to $A \cup \{x\}$, so we are indeed extending f maximally.

We extend f to $\tilde{A}$ in the following way. For each $x \in \tilde{A}$, consider the sequence $(\operatorname{cl}(f(B(x, 2^{-n}))) \subseteq Y, n \in \mathbb{N})$. This is a descending sequence of closed subsets of Y, which are inhabited because $x \in \operatorname{cl}(A)$ so $B(x, 2^{-n})$ always intersects A, and which have diameter tending to zero because $\operatorname{osc}_f(x) = 0$, so the intersection is a singleton, and we let $\tilde{f}(x)$ be its unique element.

The extended function $\tilde{f} : \tilde{A} \to Y$ is continuous, because for $U \subseteq \tilde{A}$ open, we have $\tilde{f}(U) \subseteq \mathrm{cl}(f(U))$, so $\mathrm{diam}\big(\tilde{f}(U)\big) = \mathrm{diam}(f(U))$, thus $\mathrm{osc}_{\tilde{f}} = \mathrm{osc}_f = 0$.

Finally, $\tilde{A}$ is G_δ because

$$\tilde{A} = \mathrm{cl}(A) \cap \bigcap_n \{x \in X \mid \mathrm{osc}_f(x) < 2^{-n}\}$$

and $\{x \in X \mid \mathrm{osc}_f(x) < \varepsilon\}$ is open for all $\varepsilon > 0$. □

(4.6) Let X be a completely metrizable space and let $Y \subseteq X$ be a completely metrizable subspace. Then Y is G_δ.

Proof. Kuratowski's extension theorem applied to $\mathrm{id} : Y \to Y$ gives a continuous extension $\tilde{\mathrm{id}} : \tilde{Y} \to Y$ where $Y \subseteq \tilde{Y} \subseteq \mathrm{cl}(Y) \subseteq X$ and $\tilde{Y}$ is G_δ. By continuity, this extension must again be the identity, so $Y = \tilde{Y}$. □

(4.7) All in all, combining ¶ 4.2 and ¶ 4.6, a subspace of a Polish space is Polish iff it is G_δ.

5. The Cantor space

We take a closer look at the Cantor space $\{0,1\}^{\mathbb{N}}$, which is one of our most fundamental spaces. In particular, we completely characterize which subsets of $\mathbb{R}$ are homeomorphic to the Cantor space, with Cantor's middle thirds set being the classical example.

(5.1) The Cantor space $\{0,1\}^{\mathbb{N}}$ is homeomorphic to Cantor's middle thirds set, through the map $\varphi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}, b \mapsto \sum_{n\in\mathbb{N}} 2b_n/3^{n+1}$.

Proof. The construction of Cantor's middle thirds set starts from the closed interval $[0,1]$ and iteratively removes the middle third $(a + (b-a)/3, a + 2(b-a)/3)$ from each of the intervals $[a,b]$ forming the current subset. Equivalently, the n-th step removes all reals in $[0,1]$ all of whose base 3 expansions (the proper one, and also the improper when relevant) have 1 as the n-th digit. Therefore the image of φ is indeed Cantor's middle thirds set. Moreover, φ is a continuous injection, hence a homeomorphism onto its image because the Cantor space is compact and $\mathbb{R}$ is Hausdorff. □

(5.2) More generally, a subset of $\mathbb{R}$ is homeomorphic to the Cantor space iff it is inhabited, compact, without isolated points, and has empty interior.

Proof sketch. Let $H \subseteq \mathbb{R}$. If H is homeomorphic to the Cantor space then it has empty interior because the Cantor space is totally disconnected (the three other properties only involve the topology of H itself so they are preserved by homeomorphism). Conversely, assume H has these four properties. Let a be its minimum and b its maximum, which exist because H is inhabited and compact. Consider the connected components of $[a,b] \setminus H$, which are open intervals because H is closed. They form a totally ordered set with the order induced by $\mathbb{R}$. Using Cantor's isomorphism theorem, one can verify that this totally ordered set is isomorphic to $\mathbb{Q}$, and from there one can construct a homeomorphism with Cantor's middle thirds set. □

(5.3) A topological space is ***0-dimensional*** when it has a basis of clopen sets. (In general, a topological space is (-1)-dimensional when it is empty, and α-dimensional (where α is a natural number or even an ordinal) when it has a basis of open subsets U_i such that for each i, there exists $\beta < \alpha$ such that the boundary of U_i is β-dimensional.)

(5.4) A metrizable space is homeomorphic to Cantor space iff it is inhabited, compact, without isolated points, and 0-dimensional.

Proof. TODO; for now see Kechris p. 35. □

(For general topological spaces, one can prove the following characterization: inhabited, compact, Hausdorff, second-countable, 0-dimensional, without isolated points.)

6. The Baire space

The Baire space is also a fundamental space, which we can also embed into $\mathbb{R}$, as well as in the Cantor space.

(6.1) The Baire space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Proof sketch. $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $(1, +\infty) \cap \mathbb{Q}^c$, which is homeomorphic to $(\mathbb{N}^*)^{\mathbb{N}}$ (where $\mathbb{N}^*$ denotes $\mathbb{N} \setminus \{0\}$) via mapping x to the sequence (a_n) from the continued fraction expansion

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

□

Alternative proof sketch. We recursively construct a family of open intervals $(I_s, s \in \mathbb{Z}^\star)$ as follows. We set $I_\varepsilon := \mathbb{R}$. Ater having defined $I_s := (\ell, r)$, we recursively pick rationals $\cdots < q_{-1} < q_0 < q_1 < \cdots$ in I_s so that $\inf_n q_n = \ell$ and $\sup_n q_n = r$ and $q_{n+1} - q_n < 2^{-|s|}$ and all rationals from I_s with denominator at most n are among the q_i; then we set $I_{sn} := (q_n, q_{n+1})$. The map $\mathbb{Z}^{\mathbb{N}} \to \mathbb{R} \setminus \mathbb{Q}$ sending s to the unique element of $\bigcap_n I_{s[:n]}$ is a homeomorphism. □

Yet another proof sketch. We equip $(\mathbb{Z}^*)^\star$ with the total order defined by $s < t$ when $s \cdot (0, 0, ...) < t \cdot (0, 0, ...)$ lexicographically. (If we view $(\mathbb{Z}^*)^\star$ as a tree drawn vertically, where the root is at the top and the children nodes $s \cdot n$ of s are arranged on a line below s, those with $n < 0$ on the left of s and those with $n > 0$ on the right, then we have ordered the nodes from left to right.) We have an isomorphism of totally ordered sets $\varphi : (\mathbb{Z}^*)^\star \xrightarrow{\sim} \mathbb{Q}$ from Cantor's isomorphism theorem, since $(\mathbb{Z}^*)^\star$ is countable, dense and unbounded. We define $\psi : (\mathbb{Z}^*)^{\mathbb{N}} \to \mathbb{R} \setminus \mathbb{Q}$ by letting $\psi(u)$ be the irrational whose Dedekind cut is (L, R) (i.e., $L = \{q \in \mathbb{Q} \mid q < \psi(s)\}$ and $R = \{q \in \mathbb{Q} \mid q > \psi(s)\}$), where $L := \varphi(\{s \mid s \cdot (0, 0, ...) < t\})$ and $R := \varphi(\{s \mid s \cdot (0, 0, ...) > t\})$, and we verify that ψ is a homeomorphism. □

(6.2) The Baire space embeds into the Cantor space as a G_δ subset.

Proof. Consider the map $\varphi : \mathbb{N}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ that encodes a sequence of natural numbers in binary by encoding each natural number in unary:

$$\varphi(u) = \underbrace{0...0}_{u_0 \text{ times}} 1 \underbrace{0...0}_{u_1 \text{ times}} 1...$$

This is a homeomorphism onto its image, which is the subset of those binary sequences which contain infinitely many 1s. We will later see that a Polish subspace of a Polish space is automatically G_δ (¶ 4.6), but in this case it can be seen completely explicitly: this subset is the intersection over $n \in \mathbb{N}$ of the open subset of those binary sequences which contain a 1 after the n-th term. □

Conversely, the inclusion map $\{0, 1\}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ embeds the Cantor space as a (closed) subset of the Baire space (we will generalize this in ¶ 11.2).

7. The space of compact subsets

(7.1) Let X be a metric space, usually $[0,1]$. We define the space $\mathcal{K}(X)$ as consisting of the inhabited compact subsets of X, endowed with the Hausdorff metric:

$$d_H(K,L) \coloneqq \min\{\varepsilon \geq 0 \mid L \subseteq B(K,\varepsilon) \wedge K \subseteq B(L,\varepsilon)\}.$$

The topology of $\mathcal{K}(X)$ only depends on the topology of X (not on the particular choice of metric), and if X is Polish, then $\mathcal{K}(X)$ is Polish.

Proof. The topology of $\mathcal{K}(X)$ is independent of the metric on X because it coincides with the Vietoris topology, which by definition is generated by the following subbasic opens, for each open $U \subseteq X$:

$$C(U) \coloneqq \{K \in \mathcal{K}(X) \mid K \subseteq U\},$$
$$I(U) \coloneqq \{K \in \mathcal{K}(X) \mid K \text{ intersects } U\}.$$

To see that $C(U)$ is open for the Hausdorff metric, observe that if $K \subseteq U$ then $\varepsilon > 0$ where $\varepsilon \coloneqq d(K, U^c)$ (otherwise, there are points in K arbitrarily close to U^c, thus by compactness of K there is one point in K arbitrarily close to U^c, and therefore also in U^c since it is closed, a contradiction), and then any $L \subseteq B(K,\varepsilon)$ must be contained in U. Likewise, for $I(U)$, if $x \in K \cap U$, say $x \in B(U,\varepsilon)$, then L intersects U whenever $K \subseteq B(L,\varepsilon)$. Conversely, let us check that opens for the Hausdorff metric are open in the Vietoris topology. Let $K \in \mathcal{K}(X)$ and $\varepsilon > 0$. Let $U \coloneqq B(K,\varepsilon)$ and let $V_0, \ldots, V_{n-1}$ be opens of X such that their restrictions to K form a finite $\varepsilon/2$-net in K. Then $d_H(K,L) \leq \varepsilon$ for all $L \in C(U) \cap I(V_0) \cap \cdots \cap I(V_{n-1})$.

Now assume that X is Polish. For separability of $\mathcal{K}(X)$, let D be a countable dense subset of X. We claim that the finite inhabited subsets of D form a countable dense subset of $\mathcal{K}(X)$. Let $\varepsilon > 0$ and $K \in \mathcal{K}(X)$. Define $D' \coloneqq \{x \in D \mid d(x,K) \leq \varepsilon\}$. The cover $(B(x,\varepsilon), x \in D')$ of K has a finite (inhabited) subcover $B(x_0,\varepsilon), \ldots, B(x_k,\varepsilon)$ where $x_0, \ldots, x_k \in D$. Then $d_H(K, \{x_0, \ldots, x_k\}) \leq \varepsilon$.

Completeness: for now see Kechris p. 26. □

8. The space of continuous functions

(8.1) Let K be a compact metrizable space and Y a Polish space. We endow the space of continuous functions $\mathcal{C}(K,Y)$ with the metric $d(f,g) \coloneqq \max_{x \in K} d(f(x), g(x))$. The topology of $\mathcal{C}(K,Y)$ only depends on the topology and not the choice of metric on Y, and $\mathcal{C}(K,Y)$ is Polish. (We write just $\mathcal{C}(K)$ for $\mathcal{C}(K,\mathbb{R})$.)

Proof. TODO; for now see Kechris p. 24. Heine's theorem is used for separability. For the specific case of $\mathcal{C}([0,1])$, polynomial functions with rational coefficients form a countable dense subset by Weierstrass' approximation theorem. □

9. Universal Polish spaces

Many problems about arbitrary Polish spaces can be reduced to well-known Polish spaces through universality properties.

(9.1) Every Polish space X embeds as a closed subspace of $\mathbb{R}^{\mathbb{N}}$.

Proof. We define an embedding $\varphi : X \hookrightarrow \mathbb{R}^{\mathbb{N}}$ by $\varphi(x) \coloneqq (d(x,a_0), d(x,a_1), \ldots)$, where (a_n) is a fixed countable dense family. This map is continuous and injective; let us check continuity of its inverse defined on its image. Fix $x \in X$ and $\varepsilon > 0$. We need to check that there exists a neighborhood V of $\varphi(x)$ in $\mathbb{R}^{\mathbb{N}}$ such that for all $y \in X$, if $\varphi(y) \in V$, then $y \in B(x, \varepsilon)$. Let n be such that $d(x, a_n) < \varepsilon/2$. Then $V := \{s \in [0,1]^{\mathbb{N}} \mid s_n < \varepsilon/2\}$ is such a neighborhood.

As X is Polish, it is G_δ in $\mathbb{R}^{\mathbb{N}}$ (¶ 4.6), and from there we can embed X as a closed subspace of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ (¶ 4.2), which is homeomorphic to $\mathbb{R}^{\mathbb{N}}$. □

(9.2) Every separable metrizable space X embeds into Hilbert's cube $[0,1]^{\mathbb{N}}$. In particular, every compact metrizable space embeds as a closed subspace of the Hilbert cube.

Proof. X embeds into $\mathbb{R}^{\mathbb{N}} \cong (0,1)^{\mathbb{N}}$, which embeds into $[0,1]^{\mathbb{N}}$. □

(9.3) Every inhabited compact metrizable space X is a continuous image of the Cantor space $\{0,1\}^{\mathbb{N}}$.

Proof. We recursively construct a family of inhabited closed subsets indexed by finite binary words, $(F_s \subseteq X, s \in \{0,1\}^\star)$. We initially set $F_\varepsilon := X$. Once F_s is defined, we use compactness to find finitely many closed subsets of diameter at most $2^{-|s|}$ contained in F_s and together covering F_s. After perhaps adding repetitions, we have 2^k of them for some $k \in \mathbb{N}^*$, and we define the $F_{s \cdot t}$ for $t \in \{0,1\}^k$ to be these.

For each $u \in \{0,1\}^{\mathbb{N}}$, the sequence $\left(F_{u[:n]}, n \in \mathbb{N}\right)$ is a descending sequence of inhabited closed subsets with diameters converging to zero. By completeness, its intersection contains a unique point, and we define $f(u)$ to be this point. The resulting function $f : \{0,1\}^{\mathbb{N}} \to X$ is surjective by construction, and continuous because the diameter of $F_{p[:n]}$ converges to zero. □

(9.4) Every inhabited Polish space is a continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$.

Proof. Similar to the previous, with countable instead of branching since we only have separability and not compactness. □

The converse unfortunately fails: $\mathbb{Q}$ is a continuous image of the Baire space (send (u_n) to $q(u_0)$ where $q : \mathbb{N} \to \mathbb{Q}$ is some surjection), but not completely metrizable, as we will see in ¶ 15.6. Nevertheless, this theorem allows to reduce many problems to manipulation of countably branching trees.

(9.5) *Digression.* A continuous image of $[0,1]$ is called a *Peano continuum.* For example, $[0,1]^2$ is a Peano continuum: a continuous surjection $f : [0,1] \to [0,1]^2$ is called a *space-filling curve* and there are various ways to construct one, such as the following. A first idea might be to define f through binary expansions by setting "$f(0.d_0d_1d_2d_3...) := (0.d_0d_2..., 0.d_1d_3...)$", but this does not give the same result on the proper and the improper binary expansions of a dyadic number. To fix this idea, we embed the Cantor space as the usual ternary Cantor set in $[0,1]$, we let f send the point in the Cantor set associated to a binary sequence $d_0d_1d_2d_3...$ to $(0.d_0d_2..., 0.d_1d_3...)$, and we extend f by affine parts on the connected components of the complement of the Cantor set in $[0,1]$ (the intervals that are removed at each stage of the construction).

More generally, we admit the Hahn–Mazurkiewicz theorem: a Hausdorff space is a Peano continuum iff it is inhabited, second-countable, connected, and locally connected (has a basis of connected subsets).

10. The Cantor–Bendixson theorem

We sometimes need to discard isolated points from a subset. However, this might leave new isolated points: consider $\{0\} \cup \{1/n, n \in \mathbb{N}^*\}$. How to arrive at a subset without isolated points and how much can this leave out? The answer is provided by the Cantor–Bendixson theorem.

(10.1) A subset of a separable metrizable space has only countably many isolated points.

Proof. For each isolated point x, choose a ball centered on x which meets the subset only at x. After halving the radii of all these balls, they are pairwise disjoint, so the countable chain condition applies. □

(10.2) In a separable metrizable space, every strictly increasing or strictly decreasing transfinite sequence of open or closed subsets has countable length.

Proof. Fix a countable basis. Going from one term of a strictly increasing sequence of open subsets to the next adds at least one basic open to those which are contained in the current subset. The case of a strictly decreasing sequence of open subsets is similar. (For strictly increasing sequences of opens, we could also use hereditary Lindelöfness.) The statements for closed subsets are dual. □

(10.3) A topological space is ***perfect*** when it has no isolated points. A ***perfect subset*** of a topological space is a *closed* subset without isolated points.

The terminology is heterological (not perfect): a perfect subset is *not* just a subset which is perfect as a subspace — it also needs to be closed.

(10.4) The **Cantor-Bendixson theorem** states that every Polish space X can be uniquely partitioned into a perfect subset P, called the ***perfect kernel*** of X, and a countable subset C.

In the rest of this section, we study two different constructions of the perfect kernel, which are both useful in their own right: iteratively, by removing isolated points, or in one go, by removing points around which X is countable. The uniqueness part is deferred to ¶ 11.4.

(10.5) In the first construction, we define a descending transfinite sequence of closed subsets (F_α) as follows: we let $F_0 := X$; we let $F_{\alpha+1}$ be F_α minus its isolated points (this is called the Cantor–Bendixson derivative of F_α); and we let $F_\alpha := \bigcap_{\beta<\alpha} F_\beta$ for limit α. The sequence must stabilize at a countable ordinal, since until stabilization it is a strictly decreasing sequence of closed subsets (¶ 10.2). This ordinal λ is called the ***Cantor–Bendixson rank*** of F. The subset F_λ is perfect by construction, and the remainder $X \setminus F_\lambda$ is countable because there are countably many steps, and each step removes countably many points (¶ 10.1).

(10.6) In the second construction, we define P as the set of points of X all of whose neighborhoods are uncountable, called the ***condensation points*** of X. Then P is perfect and $C := P^c$ is countable.

Proof. Observe that C is the union of all countable open subsets. In particular, it is open, so P is closed, and moreover, this cover has a countable subcover by Lindelöfness, so C is countable. If P had an isolated point x, there would be a neighborhood $U \subseteq X$ of x such that $U \cap P = \{x\}$, and then U would be countable since the points other than x come from C, contradicting $x \in P$. □

(10.7) The transfinite process can indeed reach any countable ordinal α: there exists a countable closed subset of $\mathbb{R}$ having Cantor-Bendixson rank α.

Proof. We construct such a subset H_α by induction on α. We let $H_0 \coloneqq \varnothing$. For a successor $\alpha + 1$, we let $H_{\alpha+1}$ consist of the point 0 and, for each n, a copy of H_α in the interval $\left(2^{-(n+1)}, 2^{-n}\right)$ (using a homeomorphism between $\mathbb{R}$ and this interval). For a limit α, which we can write as $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$ where $\alpha_n < \alpha$, we let H_α consist of a copy of H_{α_n} in each interval $(n, n + 1)$. □

11. Perfect schemes

In § 9, we showed that every inhabited Polish space is a continuous image of the Baire space and embeds into $\mathbb{R}^\mathbb{N}$. Here, we prove a new universality property: every uncountable Polish space embeds the Baire space and the Cantor space. As consequences, we obtain the continuum hypothesis for Polish space, namely that the cardinality of a Polish space is countable or continuum, as well as the uniqueness part of the Cantor–Bendixson theorem.

(11.1) Every inhabited perfect Polish space X embeds the Cantor space.

Proof. Recall that $\{0, 1\}^\star$ denotes the set of finite binary sequences. We are going construct a family of subsets $(F_s, s \in \{0, 1\}^\star)$ such that:

- Each F_s is closed.
- Whenever two distinct s and t have the same length, F_s and F_t are disjoint.
- Whenever s is a prefix of t, we have $F_s \subseteq F_t$.
- For every $p \in \{0, 1\}^\mathbb{N}$, we have $\operatorname{diam}\left(F_{p[:n]}\right) \to 0$.

This is called a ***perfect scheme***.

In other words, if view $\{0, 1\}^\star$ as a tree, the “Cantor tree”, a perfect scheme is a labeling of each node with a closed subset so that the labels of all nodes on the same level are disjoint, and the labels along every infinite path in the tree form a descending sequence with diameter converging to zero. In particular, by completeness of X, the intersection of all the labels along an infinite path p has a unique point $\varphi(x)$. The resulting function $\varphi : \{0, 1\}^\mathbb{N} \to X$ is continuous because of the condition that diameters converge to zero on every infinite path, so it is a homeomorphism onto its image because the Cantor space is compact and X is Hausdorff.

To construct a perfect scheme (F_s), we choose a point x_ε arbitrarily and let F_ε be the closed ball of radius 1 around x. As F_ε is uncountable, we can choose two distinct points $x_0, x_1 \in F_\varepsilon$. We let F_0 and F_1 be closed balls around x_0 and x_1 of radius under 1/2 and small enough that F_0 and F_1 are disjoint. We repeat the same construction inside F_0 and F_1, recursively. □

(11.2) As a corollary, every uncountable Polish space X embeds the Cantor space (and therefore also the Baire space, ¶ 6.2).

Proof. Let Piotr be a perfect kernel of X obtained from the Cantor–Bendixson theorem (¶ 10.4). Being closed, Piotr is Polish (despite Englishness expectations), and it is inhabited because it has countable complement and X is uncountable. Therefore Piotr embeds the Cantor space (¶ 11.1), and so does X. □

(11.3) We deduce the ***continuum hypothesis for Polish spaces***: every Polish space is countable or continuum-sized.

Proof. An uncountable Polish space X has cardinality at least continuum because it embeds the Cantor space (¶ 11.2), and at most continuum because it embeds into $\mathbb{R}^{\mathbb{N}}$ (¶ 9.1). $\square$

(11.4) Separately, we obtain the promised uniqueness in the Cantor–Bendixson theorem (¶ 10.4).

Proof. Let X be Polish and let $X = P' \cup C'$ be a partition of X into a perfect subset P and a countable subset C. We are going to prove that this decomposition coincides with the decomposition $X = P \cup C$ constructed in ¶ 10.6, where P consists of the condensation points of X. First, every $x \in C'$ has C' as a countable neighborhood, so $C' \subseteq C$. To prove $P' \subseteq P$, let $x \in P'$ and let $U \subseteq X$ be a neighborhood of x. We need to show that U is uncountable. The subspace $P \cap U$ is G_δ hence Polish (¶ 4.2), inhabited by x, and perfect because P is perfect and $P \cap U$ is open in P. Being an inhabited perfect Polish space, $P \cap U$ embeds the Cantor space (¶ 11.2), hence $P \cap U$ is uncountable and so is U. $\square$

12. The Borel hierarchy

The Borel subsets of a topological space are generated from open subsets by closing under complementation and countable unions. We investigate how Borel subsets are stratified by the stages of this inductive construction. We have already met F_σ and G_δ subsets, which are the next level of complexity, and there are transfinitely many more levels beyond.

(12.1) Let X be a topological space. The ***Borel hierarchy*** on X is a hierarchy of classes of subsets $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{P}(X)$ and $\mathbf{\Pi}^0_\alpha(X) \subseteq \mathcal{P}(X)$ indexed by non-zero countable ordinals α.[5] A member of $\mathbf{\Sigma}^0_\alpha(X)$ (resp. $\mathbf{\Pi}^0_\alpha(X)$) is called "a $\mathbf{\Sigma}^0_\alpha$ subset of X" (resp. "a $\mathbf{\Pi}^0_\alpha$ subset of X").

- A subset is $\mathbf{\Sigma}^0_1$ when it is open, and $\mathbf{\Pi}^0_1$ when it is closed.
- For $1 < \alpha < \omega_1$, a subset is $\mathbf{\Sigma}^0_\alpha$ when it is a countable union of subsets each of which is $\mathbf{\Pi}^0_{<\alpha}$, i.e., $\mathbf{\Sigma}^0_\beta$ for some $\beta < \alpha$ (depending on the subset), and $\mathbf{\Pi}^0_\alpha$ when it is a countable intersection of $\mathbf{\Sigma}^0_{<\alpha}$ subsets.

In particular, $\mathbf{\Sigma}^0_2$ means F_σ and $\mathbf{\Pi}^0_2$ means G_δ.

We also define a subset to be $\mathbf{\Delta}^0_\alpha$ when it is both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$. In particular, $\mathbf{\Delta}^0_1$ means clopen. The $\mathbf{\Delta}^0_2$ subsets are those which are both G_δ and F_σ, and are called ambiguous subsets. More generally, $\mathbf{\Delta}^0_\alpha$ subsets are called α-ambiguous subsets.

(12.2) The following basic properties of the Borel hierarchy are immediate:

- The $\mathbf{\Sigma}^0_\alpha$ subsets are exactly the complements of the $\mathbf{\Pi}^0_\alpha$ subsets.
- For $\alpha < \beta$, every $\mathbf{\Sigma}^0_\alpha$ subset is $\mathbf{\Pi}^0_\beta$ and every $\mathbf{\Pi}^0_\alpha$ subset is $\mathbf{\Sigma}^0_\alpha$.
- Moreover, assuming if $\alpha = 1$ and $\beta = 2$ that X is metrizable, every $\mathbf{\Sigma}^0_\alpha$ subset is $\mathbf{\Sigma}^0_\beta$ and every $\mathbf{\Pi}^0_\alpha$ subset is $\mathbf{\Pi}^0_\beta$. (For $\alpha = 1$ and $\beta = 2$, this means that open subsets are F_σ and closed subsets are G_δ, which we remarked in ¶ 4.1.)
- The Borel sets are exactly those which are $\mathbf{\Sigma}^0_\alpha$ for some α, or equivalently $\mathbf{\Pi}^0_\alpha$ for some α.
- The $\mathbf{\Sigma}^0_\alpha$ subsets are closed under countable unions and finite intersections, while the $\mathbf{\Pi}^0_\alpha$ subsets are closed under countable intersections and finite unions. The $\mathbf{\Delta}^0_\alpha$ subsets are closed under finite unions, finite intersections and complements.

[5]The $\mathbf{\Sigma}$ and $\mathbf{\Pi}$ are in bold to distinguish this "boldface" hierarchy from the "lightface" Borel hierarchy studied in effective descriptive set theory, where concepts from classical descriptive set theory find computability-theoretic counterparts.

- Being $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$ is preserved by continuous preimages.
- The $\boldsymbol{\Sigma}^0_\alpha$ subsets of a subspace are the intersections with the subspace of $\boldsymbol{\Sigma}^0_\alpha$ subsets of the whole space (in particular, a $\boldsymbol{\Sigma}^0_\alpha$ subset of a $\boldsymbol{\Sigma}^0_\alpha$ subset is $\boldsymbol{\Sigma}^0_\alpha$), and likewise for $\boldsymbol{\Pi}^0_\alpha$. On the other hand, this is very false for $\boldsymbol{\Delta}^0_\alpha$ (take the space $\mathbb{R}$ with the subspace $[0,1] \cup [2,3]$, where $[0,1]$ is clopen).
- In a countable Hausdorff space, the Borel hierarchy is trivialized: as singletons are closed, every subset is F_σ (and thus every subset is G_δ as well).

(12.3) If a topological space has at most κ open subsets, where κ is an uncountable cardinal, then it has at most $\kappa^{\aleph_0}$ Borel subsets (where $\aleph_0$ denotes the countably infinite cardinal). In particular, if it has at most continuum-many open subsets, then it has at most continuum-many Borel subsets.

Proof. By an immediate induction, each level of the Borel hierarchy contains at most $\kappa^{\aleph_0}$ subsets. Summing all the levels, this makes at most $\aleph_1 \times \kappa^{\aleph_0}$ Borel subsets, and $\aleph_1 \times \kappa^{\aleph_0} = \kappa^{\aleph_0}$ since $\aleph_1 \leq \kappa \leq \kappa^{\aleph_0}$. In the particular case $\kappa = 2^{\aleph_0}$, we have $\kappa^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$. □

(12.4) An infinite Polish space X has exactly continuum-many open subsets, and hence exactly continuum-many Borel subsets (¶ 12.3).

Proof. If X is discrete, all subsets are open. Otherwise, there is a non-isolated point x. Pick a sequence of distinct points (x_n) converging to x. Then $P \mapsto \{x_n, n \in P\} \cup \{x\}$ is an injection from $\mathcal{P}(\mathbb{N})$ into the closed subsets of X. In both cases, X has at least continuum-many open subsets. The converse inequality holds because X is second-countable. □

13. Universal subsets for levels of the Borel hierarchy

Our next goal is to prove that the Borel hierarchy on an uncountable Polish space is strict. The proof is a diagonalization for which we first need to construct universal subsets, which are interesting in their own right.

(13.1) A ***pointclass*** Γ is the data of a class of subsets $\Gamma(X) \subseteq \mathcal{P}(X)$ for each Polish X, thought of as a "level of topological complexity". Examples are $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$, and Borel subsets. An element of $\Gamma(X)$ is often called "a Γ subset of X".

The ***dual*** pointclass of Γ is the pointclass whose subsets are the complements of Γ subsets (e.g., the dual of $\boldsymbol{\Sigma}^0_\alpha$ is $\boldsymbol{\Pi}^0_\alpha$).

(13.2) Let X and Y be Polish and let Γ be a pointclass. A ***universal $\boldsymbol{\Gamma}$ subset*** for X and Y, or more informally a universal Γ subset of $X \times Y$, is a Γ subset U of $X \times Y$ such that every Γ subset A of X can be written as the ***section*** of Y at some $y \in Y$, namely $U^y := \{x \in X \mid (x,y) \in U\}$ of U at some $y \in Y$. (We do not require that all sections of U are Γ subsets, only that every Γ subset is a section of U.)

Note that a universal subset for the dual of Γ is just the complement of a universal Γ subset.

(13.3) Let X be Polish and Y uncountable Polish and let $1 \leq \alpha < \omega_1$. There exist universal $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$ subsets of $X \times Y$.

Proof. We can assume wlog that Y is the Cantor space $\{0,1\}^{\mathbb{N}}$. Indeed, we can embed the Cantor space into Y (¶ 11.2), and assuming we already have a universal $\boldsymbol{\Pi}^0_\alpha$ subset U of $X \times \{0,1\}^{\mathbb{N}} \subseteq X \times Y$, the same U is a universal $\boldsymbol{\Pi}^0_\alpha$ subset of $X \times Y$: it is $\boldsymbol{\Pi}^0_\alpha$ in $X \times Y$

because $\{0,1\}^{\mathbb{N}}$ is closed (being compact) hence $\mathbf{\Pi}^0_\alpha$ in Y, and every $\mathbf{\Pi}^0_\alpha$ subset of X is indeed a section of it.

For $\alpha = 1$, let (B_n) be a countable basis of X. Then $\bigcup_n B_n \times \{s \in 2^{\mathbb{N}} \mid s_n = 1\}$ is a universal open subset of $X \times \{0,1\}^{\mathbb{N}}$.

For $\alpha \geq 2$, we proceed by transfinite induction. Using the homeomorphism $2^{\mathbb{N}} \cong (2^{\mathbb{N}})^{\mathbb{N}\times\{\beta \mid \beta<\alpha\}}$, it suffices to exhibit a universal $\mathbf{\Sigma}^0_\alpha$ subset U of $X \times (2^{\mathbb{N}})^{\mathbb{N}\times\{\beta \mid \beta<\alpha\}}$. By induction hypothesis, we have for each $\beta < \alpha$ a universal $\mathbf{\Pi}^0_{\beta_n}$ subset U_β of $X \times 2^{\mathbb{N}}$, and we take $U \coloneqq \bigcup_{n\in\mathbb{N}} \bigcup_{\beta<\alpha} C_{n,\beta}$ where

$$C_{n,\beta} \coloneqq \left\{(x,s) \in X \times (2^{\mathbb{N}})^{\mathbb{N}\times\{\beta \mid \beta<\alpha\}} \mid (x, s(n,\beta)) \in U_\beta\right\}.$$

Observe that $C_{n,\beta}$ is the preimage of U_β by the continuous function $(x,s) \mapsto (x, s(n,\beta))$, hence $\mathbf{\Pi}^0_\beta$, so U is $\mathbf{\Sigma}^0_\alpha$. Now take a $\mathbf{\Sigma}^0_\alpha$ subset A of X. By writing it as a countable union of $\mathbf{\Pi}^0_{<\alpha}$ subsets and inserting empty subsets where needed, we can write

$$A = \bigcup_n \bigcup_{\beta<\alpha} A_{n,\beta}$$

where each $A_{n,\beta}$ is $\mathbf{\Pi}^0_\beta$, hence equals the section $(U_\beta)^{s(n,\beta)}$ for some $s(n,\beta) \in \{0,1\}^{\mathbb{N}}$. Then A is the section U^s. □

(13.4) Let X be an uncountable Polish space. There exists a $\mathbf{\Sigma}^0_\alpha$ subset $H \subseteq X$ which is not $\mathbf{\Pi}^0_\alpha$ (and its complement is $\mathbf{\Pi}^0_\alpha$ and not $\mathbf{\Sigma}^0_\alpha$).

Proof. We use a diagonal argument. Let U be a universal $\mathbf{\Pi}^0_\alpha$ subset of $X \times X$ and let $H \coloneqq \{x \in X \mid (x,x) \notin U\}$. Then H is $\mathbf{\Sigma}^0_\alpha$ as the continuous preimage by $x \mapsto (x,x)$ of U^c. Assuming H is $\mathbf{\Pi}^0_\alpha$, there exists a $y \in X$ such that $H = U^y$, and then $y \in H \Leftrightarrow (y,y) \notin U \Leftrightarrow y \notin U^y \Leftrightarrow y \notin H$, a contradiction. □

We conclude that the Borel hierarchy on an uncountable Polish space X is completely strict. The inclusion $\mathbf{\Delta}^0_\alpha \subseteq \mathbf{\Sigma}^0_\alpha$ is strict because H is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$, hence not $\mathbf{\Delta}^0_\alpha$, and likewise $\mathbf{\Delta}^0_\alpha \subseteq \mathbf{\Pi}^0_\alpha$ is strict because of H^c. The inclusion $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$ is strict because H^c is $\mathbf{\Pi}^0_\alpha$, hence $\mathbf{\Delta}^0_{\alpha+1}$, but not $\mathbf{\Sigma}^0_\alpha$, and likewise $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$ is strict because of H.

(13.5) On the other hand, there is never a universal Δ^0_α subset of $X \times X$ for any topological space X and non-zero countable ordinal α, and likewise there is no universal Borel subset of $X \times X$.

Proof. We use a diagonal argument again. Assume $U \subseteq X \times X$ is universal Δ^0_α, and let $H \coloneqq \{x \in X \mid (x,x) \notin U\}$. Then H is Δ^0_α since complements and continuous preimages of Δ^0_α subsets are Δ^0_α, so there exists y such that $A = U^y$, and then $y \in A \Leftrightarrow (y,y) \in U \Leftrightarrow y \notin A$, contradiction. The argument is the same for Borel subsets. □

14. Structural properties of Borel classes

(14.1) Let X be a set. A class of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ has the ***reduction property*** when for all $A, B \in \mathcal{S}$, there exists a partition of $A \cup B$ into two subsets $A', B' \in \mathcal{S}$ such that $A' \subseteq A$ and $B' \subseteq B$. It has the ***separation property*** when for all disjoint $A, B \in \mathcal{S}$, there exists $C \subseteq X$ such that $C \in \mathcal{S}$, $C^c \in \mathcal{S}$, $A \subseteq C$ and $B \subseteq C^c$. If $\mathcal{S}$ has reduction, then the dual pointclass of $\mathcal{S}$ has separation.

(14.2) For all $2 \leq \alpha < \omega_1$, the pointclass $\mathbf{\Sigma}^0_\alpha$ has the reduction property (meaning that $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{P}(X)$ has the reduction property for every Polish X). In particular, $\mathbf{\Pi}^0_\alpha$ has the separation property.

Proof. TODO; for now see Kechris p. 172 □

(14.3) Let X be an uncountable Polish space and let $2 \leq \alpha < \omega_1$. The class $\mathbf{\Pi}^0_\alpha(X)$ does not have the reduction property. In particular, $\mathbf{\Sigma}^0_\alpha(X)$ does not have the separation property.

Proof. Let U be a universal $\mathbf{\Pi}^0_\alpha$ subset of $X \times \{0,1\}^{\mathbb{N}}$. Let

$$U_0 := \{(x,u) \mid (x, \varphi_0(u)) \in U\} \subseteq X \times \{0,1\}^{\mathbb{N}}$$
$$U_1 := \{(x,u) \mid (x, \varphi_1(u)) \in U\} \subseteq X \times \{0,1\}^{\mathbb{N}}$$

where φ_0, φ_1 are the two components of a homeomorphism $\{0,1\}^{\mathbb{N}} \cong (\{0,1\}^{\mathbb{N}})^2$. Observe that U_0 and U_1 form a ***universal pair*** of $\mathbf{\Pi}^0_\alpha$ subsets of $X \times \{0,1\}^{\mathbb{N}}$, which means that they are $\mathbf{\Pi}^0_\alpha$ in $X \times \{0,1\}^{\mathbb{N}}$ and for every $\mathbf{\Pi}^0_\alpha$ subsets A, B of X, there exists a common $u \in \{0,1\}^{\mathbb{N}}$ such that $A = U_0^u$ and $B = U_1^u$. Indeed, U_0 and U_1 are $\mathbf{\Pi}^0_\alpha$ are continuous preimages of U, and for all $\mathbf{\Pi}^0_\alpha$ subsets $A, B \subseteq X$, we have $u_0, u_1 \in \{0,1\}^{\mathbb{N}}$ such that $A = U^{u_0}$ and $B = U^{u_1}$, and then $A = U_0^u$ and $B = U_1^u$ where $u \in \{0,1\}^{\mathbb{N}}$ is unique such that $\varphi_0(u) = u_0$ and $\varphi_1(u) = u_1$.

Let us apply the reduction property of $\mathbf{\Pi}^0_\alpha$ to U_0 and U_1, obtaining U_0' and U_1'. We claim that U_0' and U_1' cannot be separated by a $\mathbf{\Delta}^0_\alpha$ subset $D \subseteq X \times \{0,1\}^{\mathbb{N}}$. Assume towards a contradiction that D is such.

We prove that D is a $\mathbf{\Delta}^0_\alpha$-universal subset of $X \times \{0,1\}^{\mathbb{N}}$. Let $A \subseteq X$ be $\mathbf{\Delta}^0_\alpha$. As U_0 and U_1 form a universal $\mathbf{\Pi}^0_\alpha$ pair, there is a $u \in \{0,1\}^{\mathbb{N}}$ such that $A = U_0^u$ and $A^c = U_1^u$. Then $A = U_0'^u$ and $A^c = U_1'^u$: changing U_0 to U_0' and U_1 to U_1' makes no difference because U_0^u and U_1^u are disjoint and the reduction only affects points in the intersection of U_0 and U_1. As $U_0'^u$ and $U_1'^u$ cover X, the separation step does not affect this either, so $D^u = A$.

This entails a contradiction because we can embed $\{0,1\}^{\mathbb{N}}$ into X, where it is closed (being compact) hence $\mathbf{\Delta}^0_\alpha$ (because $\alpha \geq 2$), so that the embedding makes D a universal $\mathbf{\Delta}^0_\alpha$ subset of $X \times X$, which we proved is impossible. □

15. Baire category

We introduce meager subsets, a notion of being "small", which is similar in many ways to null sets in measure theory, but at the same time somewhat orthogonal. Unlike measure theory, it makes sense in any topological space. Meager subsets were first studied by Baire who called them "sets of first category", hence the name "Baire category" for the theory surrounding them.

(15.1) Let X be a topological space. A subset $H \subseteq X$ is ***nowhere dense*** when it is not dense in any inhabited open subset, i.e., every inhabited open subset contains an inhabited open subset disjoint from H, or equivalently, the closure of H has empty interior. A countable union of nowhere dense subsets is said to be ***meager***. Equivalently, a meager subset is a subset of a countable union of closed subsets with empty interior.

The complements of meager subsets are called ***comeager*** (or residual). The phrase "the generic $x \in X$ satisfies the property P" means that the subset of points with the property P is comeager, just like "almost every $x \in X$ satisfies the property P" in measure theory means that the subset of points with the property P has full measure.

(15.2) By a Venn diagram chasing, being nowhere dense and being meager are preserved by expanding the space: if $H \subseteq Y \subseteq X$ and H is nowhere dense (resp. meager) as a subset of the subspace Y, then H is meager as a subset of X.

These properties are also inherited in *open* subspaces: if $U \subseteq X$ is open and $H \subseteq X$ is nowhere dense (resp. meager), then $H \cap U$ is nowhere dense (resp. meager) as a subset of the subspace U — we say that H is "nowhere dense in U" (resp. "meager in U"). (This fails for general subspaces: $\{0\}$ is nowhere dense in $\mathbb{R}$ and not meager in itself.)

(15.3) In a *perfect* Polish space (mind the hypothesis!), singletons are nowhere dense, so a countable subset is meager.

(15.4) The **Baire category theorem** asserts that in a completely metrizable space, meager subsets have empty interior. Equivalently: a countable union of closed subsets with empty interior still has empty interior; a subset is meager iff it is contained in an F_σ with empty interior. Dually: comeager subsets are dense; a countable intersection of dense opens is dense; and a subset is comeager iff it contains a dense G_δ.

Proof. Let (U_n) be a countable family of dense opens. We want to show that $\bigcap_n U_n$ is dense. It suffices to show that it intersects every ball $B(x_0, r_0)$, wlog $r \leq 1$. As U_0 is dense, the open $B(x_0, r_0) \cap U$ is inhabited by some x_1, and we can find $r_1 > 0$, wlog $r_1 \leq 1/2$, such that $\overline{B}(x_1, r_1) \subseteq B(x_0, r_0) \cap U$. We continue in the same way, giving sequences (x_n), (r_n) such that $\overline{B}(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n) \cap U_n$ and $0 < r_n < 2^{-n}$. Then $\overline{B}(x_0, r_0) \supseteq \overline{B}(x_1, r_1) \supseteq \cdots$ is nested sequence of closed subsets with diameters converging to zero, thus its intersection is inhabited by some x, and then $x \in B(x_0, r_0) \cap \bigcap_n U_n$. ☐

This provides an intuition: a comeager subset is a subset such that inside every inhabited open, we can find a point of the subset "by successive approximations" (it is dense for a particularly simple reason).

(In general, a topological space where meager subsets have empty interior is said to be a Baire space, but this terminology clashes with *the* Baire space $\mathbb{N}^\mathbb{N}$.)

(15.5) A corollary of the Baire category theorem is that an inhabited completely metrizable space is not meager as a subset of itself, so if we want to prove that there exists a point with a certain property, it suffices to prove that a generic point has this property. This is surprisingly useful. For example, a generic function $\mathcal{C}([0,1])$ is nowhere differentiable, and we deduce the existence of a continuous nowhere differentiable function.

Proof sketch. Let

$$A_n := \left\{ f \in \mathcal{C}([0,1]) \mid \exists x \in [0,1], \forall y \in [0,1] \setminus \{x\}, \left| \frac{f(y) - f(x)}{y - x} \right| \leq n \right\}.$$

Then $\bigcup_n A_n$ contains all somewhere differentiable functions. One can check that each A_n is closed (use Heine's theorem) with empty interior (use the Stone–Weierstrass theorem and add a function with small but abrupt variations). ☐

(15.6) Another consequence is that $\mathbb{Q}$ is not G_δ in $\mathbb{R}$ and therefore not a Polish space. If $\mathbb{Q}$ was G_δ, as it is dense, it would be comeager, but it is meager (being a countable subset of non-isolated points), so $\mathbb{R}$ itself would be meager, contradicting the Baire category theorem.

In particular, there is no function $\mathbb{R} \to \mathbb{R}$ which is continuous exactly at rational points, because the continuity points form a G_δ subset (¶ 4.4). (On the other hand, there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous exactly at irrational points: set $f(p/q) := 1/q$ when p/q is a reduced fraction and $f(x) = 0$ for irrational x.)

More generally, in a completely metrizable space, if a meager subset is G_δ, then it is already nowhere dense (if it was dense somewhere, then inside that inhabited open, it would be meager and comeager).

(15.7) Yet another consequence of the Baire category theorem is that a comeager subset H of an inhabited *perfect* Polish space X has cardinality continuum.

Proof. H contains a comeager G_δ subset G, which must be non-meager by the Baire category theorem, hence is uncountable because X is perfect (¶ 15.3). Being G_δ, it is Polish (¶ 4.2), therefore it must have cardinality continuum by the continuum hypothesis for Polish spaces (¶ 11.3). □

(15.8) More examples of the "generic" quantifier:

- A generic $f \in \mathcal{C}([0,1])$ is nowhere monotone, i.e., not monotone on any inhabited open interval (this is weaker but easier to show than the statement with nowhere differentiability).

  Proof. It suffices to prove that a generic $f \in \mathcal{C}([0,1])$ is nowhere weakly increasing. For all open interval $I \subseteq [0,1]$, the set A_I which are weakly increasing (resp. weakly decreasing) on I is closed with empty interior, and if f is somewhere weakly increasing then $f \in A_I$ for some I with rational endpoints. □

- A generic $K \in \mathcal{K}([0,1])$ is null (for the Lebesgue measure λ).

  Proof. It suffices to prove that each $A_n := \{K \in \mathcal{K}([0,1]) \mid \lambda(K) < 2^{-n}\}$ is dense open, since K is null iff $K \in \bigcap_n A_n$. It is dense because it contains the finite inhabited subsets, which are already dense (¶ 7.1). That it is open is a consequence of the outer regularity of the Lebesgue measure: for all $K \in A_n$, the measure $\lambda(K)$ is the infimum of $\lambda(U)$ over open supersets $U \supseteq K$, so there exists such a U with $\lambda(U) < 2^{-n}$; we have $d(K, U^c) > 0$ because K is compact and U^c closed, and $B(K, d(K, U^c)) \subseteq A_n$. □

- A generic $K \in \mathcal{K}([0,1])$ has no isolated point.

  Proof sketch. For all open interval $I \subseteq [0,1]$, the set A_I of $K \in \mathcal{K}([0,1])$ having exactly one point in I is closed with empty interior, and every $K \in \mathcal{K}([0,1])$ with an isolated point is in A_I for some I with rational endpoints. □

- A generic $K \in \mathcal{K}([0,1])$ has empty interior.

  Proof sketch. For all open interval $I \subseteq [0,1]$, the set $A_I := \{K \in \mathcal{K}([0,1]) \mid I \subseteq K\}$ is closed with empty interior, and every $K \in \mathcal{K}([0,1])$ with inhabited interior is in A_I for some I with rational endpoints. □

- A generic $K \in \mathcal{K}([0,1])$ is homeomorphic to the Cantor space.

  Proof. This follows from the two previous points and the characterization of subsets of $\mathbb{R}$ which are homeomorphic to the Cantor space from ¶ 5.2. □

- A generic $K \in \mathcal{K}([0,1])$ is linearly independent over $\mathbb{Q}$.

  Proof sketch. For each $n \in \mathbb{N}$ and not all zero $q_0, \ldots, q_{n-1} \in \mathbb{Q}^n$, the set of $K \in \mathcal{K}([0,1])$ such that $\sum_{i=0}^{n-1} q_i x_i = 0$ for some $x_0, \ldots, x_{n-1} \in K$ has empty interior by density of the finite inhabited subsets in $\mathcal{K}([0,1])$, and it is closed: its complement is open because if K is in the complement, there is an $\varepsilon > 0$ such that $\left|\sum_{i=0}^{n-1} q_i x_i\right| > \varepsilon$ for all

$x_0, \ldots, x_{n-1} \in K$ by compactness of K^n, which shows that L is in the complement for L sufficiently close to K. □

(15.9) An example application of the Baire category theorem in analysis: if a continuous function $f : [0, +\infty) \to \mathbb{R}$ is such that $f(na) \underset{n\to+\infty}{\to} 0$ for all $a > 0$, then $f(x) \underset{x\to+\infty}{\to} 0$.

Proof. Let $\varepsilon > 0$. Define

$$F_n := \{a > 0 \mid \forall k \geq n, f(ka) \leq \varepsilon\}.$$

Each F_n is closed because f is continuous, and the F_n cover $(0, +\infty)$. By the Baire category theorem, some F_n has inhabited interior, say $(b, b + \eta) \subseteq F_n$. This means that f is at most ε on the intervals $(b, b + \eta)$, $(2b, 2b + 2\eta)$, $(3b + 3\eta)$, … As these are regularly spaced and their lengths grow to infinity, their union contains some interval $(d, +\infty)$, meaning that $f(x) \leq \varepsilon$ for all $x \geq d$. □

16. The Baire property

In measure theory, a subset of $\mathbb{R}^n$ is Lebesgue-measurable when it coincides with a Borel subset outside a null subset (in the sense that it can be written as the symmetric difference of a Borel subset and a null subset). We investigate the analogous property for Baire category.

(16.1) A subset of a topological space has the ***Baire property*** when it coincides with an open subset outside a meager subset.

Beware that the decomposition is generally not unique: in $\mathbb{R}$, the subset $\mathbb{R}$ coincides with the open $\mathbb{R}$ outside the meager $\varnothing$ and also coincides with the open $\mathbb{R} \setminus \{0\}$ outside the meager $\{0\}$.

For example, a meager subset has the Baire property (it coincides with the empty subset outside itself), as does a comeager subset (it coincides the the whole space outside its complement).

(16.2) The subsets with the Baire property form a σ-algebra (namely the σ-algebra generated by the open subsets and the meager subsets). In particular, a subset which coincides with a Borel subset outside of a meager subset has the Baire property. Moreover, a subset with the Baire property dually differs from some closed subset by a meager subset.

Proof. Closure under complementation: Let H coincide with the open U outside the meager M. Then H^c coincides with U^c outside M, and the closed subset U^c coincides with its interior $\mathrm{int}(U^c)$ outside $U^c \setminus \mathrm{int}(U^c)$, which is closed with empty interior. Overall, H coincides with the open $\mathrm{int}(U^c)$ outside the meager $M \cup (U^c \setminus \mathrm{int}(U^c))$.

Closure under countable unions: Let each H_n coincide with the open U_n outside the meager M_n. Then $\bigcup_n H_n$ coincides with the open $\bigcup_n U_n$ outside the meager $\bigcup_n M_n$. □

(16.3) In a Polish space, we have the following characterization: a subset $H \subseteq X$ has the Baire property iff for a generic point $x \in X$, either H is locally meager x or it is locally comeager at x (meaning that there exists a neighborhood V of x such that H is meager or comeager in V).

Proof. Assume H has the Baire property, say H coincides with the open U outside the meager M. For $x \in U$, a neighborhood of x in which H is comeager is U itself, since $M \cap U$ is meager in X and therefore in the open subspace U. Similarly, for $x \in \mathrm{int}(U^c)$, a neighborhood of x in which H is meager is $\mathrm{int}(U^c)$. In the remaining case, x belongs to the meager subset $U^c \setminus \mathrm{int}(U^c)$.

Conversely, assume H is locally meager or comeager at a generic $x \in X$, say for $x \notin M$ where M is meager. Let U be the set of $x \in X$ such that H is locally comeager at x, which is open. We claim that H coincides with U outside a meager.

For each $x \in U$, pick a neighborhood V_x, wlog contained in U, such that H is comeager in V_x. By hereditary Lindelöfness, a countable subfamily (V_n) of these covers U. We have $U \setminus (H \cap U) = \bigcup_n V_n \setminus (H \cap V_n)$ and each $V_n \setminus (H \cap V_n)$ is meager in V_n. This shows that $H^c \cap U$ is meager.

It remains to prove that $H \cap U^c \cap M^c$ is meager. At a point of this subset, H is locally meager, so we can use the same argument. $\square$

(16.4) In a Polish space, every subset with the Baire property can be written $H \setminus M$ where H is F_σ and M is meager, and dually as $H \cup M$ where H is G_δ and M is meager.

This is a further parallel with measure theory, where a Lebesgue-measurable subset of $\mathbb{R}^n$ can be written as $H \setminus N$ where H is G_δ and N is null, or $H \cup N$ where H is F_σ and N is null — but note that F_σ and G_δ are swapped!

Proof. Let H coincide with the open U outside the meager M. We have $M \subseteq M'$ for some meager F_σ subset M' (write M as a countable union of nowhere dense subsets and let M' be the union of the closures of these). Opens are F_σ in a Polish space, so $U \cup M'$ is F_σ, and $(U \cup M') \setminus U$ is meager since it is contained in M'. $\square$

(16.5) There exist subsets of $\mathbb{R}$ without the Baire property. The simplest example is the same as for a non-Lebesgue-measurable subset: consider a ***Vitali set*** H, i.e., a choice of representatives of all equivalences classes in the additive group $\mathbb{R}$ modulo the subgroup $\mathbb{Q}$. This means that $(H + q, q \in \mathbb{Q})$ is a partition of $\mathbb{R}$. If H coincides with an open U outside a meager M, then either U is empty, in which case H and thus $\mathbb{R}$ is meager, contradicting the Baire category theorem, or U contains an inhabited open interval (a, b), so H is comeager in (a, b), but then H intersects $H + \varepsilon$ for $0 < \varepsilon < |b - a|$ by the Baire category theorem again since both are comeager in the same inhabited open interval $(a + \varepsilon, b)$, whence a contradiction by taking ε rational.

(16.6) More generally, any inhabited perfect Polish space X has a subset without the Baire property.

To prove this, we observe that if $H \subseteq X$ is a non-meager subset with the Baire property, then H contains an inhabited perfect subset of X. Indeed, by ¶ 16.4, we can write H as $G \cup M$ where G is G_δ and M is meager. As H is non-meager, G also is, therefore G is uncountable because X is perfect. As G is G_δ, it is Polish (¶ 4.2). Being an uncountable Polish space, G embeds the Cantor space (¶ 11.2), giving an inhabited perfect subset of X contained in H.

A ***Bernstein set*** in X is a subset $H \subseteq X$ such that neither H nor H^c contains an inhabited perfect subset of X. The previous argument shows that a Bernstein set cannot have the Baire property, since one of H and H^c must be non-meager by the Baire category theorem.

To construct a Bernstein set, we take a transfinite enumeration $(P_\alpha, \alpha < \kappa)$ of the inhabited perfect subsets of X, where κ is at most continuum because X is second-countable so it has at most continuum-many closed subsets (we could in fact prove that there are indeed continuum-many perfect subsets). We iteratively pick two distinct points $x_\alpha, y_\alpha \in P_\alpha$ also distinct from all previous x_β and y_β (which is possible because P_α is continuum-sized, ¶ 11.1, and there are less than continuum $\beta < \alpha$), and we put all points x_α in H and all y_α in H^c (and each remaining point at the end of the transfinite iteration is arbitrarily put in H or H^c).

(16.7) On the other hand, there are pathological uncountable Polish spaces, necessarily not perfect, where every subset has the Baire property. For example, consider the subspace $X := P \cup C$ of $\mathbb{R}^2$, where $P := \mathbb{R} \times \{0\}$ and $C := \{(p/q, 1/q), p \in \mathbb{Z}, q \in \mathbb{N}^*\}$. The points of C are isolated, so every subset of C is open, and P is closed with empty interior, so every subset of P is meager, whence every subset of X has the Baire property.

(16.8) We also define the ***Baire property for functions***: a function between topological spaces has the Baire property when the preimage of an open subset has the Baire property.

Warning: This does *not* imply that the preimage of a subset with the Baire property has the Baire property (hence the alternative terminology "Baire-measurable function" is misleading: this does not mean a function which is measurable with respect to the σ-algebras of subsets with the Baire property on the domain and the target). A counterexample is the map $\{0,1\}^{\mathbb{N}} \to \mathbb{R}, b \mapsto \sum_{n=0}^{+\infty} 2b_n/3^{n+1}$ that induces the usual homeomorphism between the Cantor space and Cantor's middle thirds set (¶ 5.1). This function has the Baire property because it is even continuous, but its image is closed with empty interior, hence meager, hence *every* subset of the image is meager (in $\mathbb{R}$), but the Cantor space is an inhabited perfect Polish space so it has a subset without the Baire property (¶ 16.6).

(16.9) Let X and Y be Polish spaces. A function $f : X \to Y$ has the Baire property iff there exists a comeager $H \subseteq X$ such that the restriction $f_{|H} : H \to Y$ is continuous.

This is an analog of Luzin's theorem in measure theory, which states that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable, there exist Lebesgue-measurable subsets $H \subseteq \mathbb{R}$ having complements of arbitrarily small Lebesgue measure such $f_{|H}$ is continuous.

Proof. Let (V_n) be a countable basis of Y. Each $f^{-1}(V_n)$ differs from some open subset by some meager subset M_n, and we can take $H := X \setminus \left(\bigcup_n M_n\right)$. □

17. The Kuratowski–Ulam theorem

The essence of Fubini's theorem is that if (X, μ) and (Y, ν) are σ-finite measure spaces and $H \subseteq X \times Y$ is a $(\mu \times \nu)$-null subset, then the section H_x is ν-null for μ-almost every x. This also has a counterpart for the Baire property.

(17.1) Kuratowski–Ulam theorem: Let X, Y be second-countable topological spaces.

(1) If a subset $H \subseteq X \times Y$ is nowhere dense, then the section H_x is nowhere dense for the generic $x \in X$. The same holds with nowhere dense replaced by meager, or by comeager, or by having the Baire property.
(2) Let X and Y be topological spaces with subsets $A \subseteq X$ and $B \subseteq Y$. Then $A \times B \subseteq X \times Y$ is meager iff A is meager or B is meager.
(3) The converse of (1) for meager subsets holds with an extra assumption: If a subset $H \subseteq X \times Y$ is such that H_x is meager for the generic $x \in X$, and additionally H has the Baire property, then H is meager.

Proof. Fix countable bases $(U_n, n \in \mathbb{N})$ of X and $(V_n, n \in \mathbb{N})$ of Y, consisting of inhabited open subsets.

(1) Let $H \subseteq X \times Y$ be nowhere dense. Let $C \subseteq X$ consist of those x such that H_x is somewhere dense. We want to show that C is comeager. We write it as $C = \bigcup_n C_n$ where $C_n \subseteq X$ consists of those x such that H_x is dense in V_n, and we claim that C_n is nowhere dense. If it was dense in some inhabited open $U \subseteq X$, then H would be dense in the inhabited open $U \times V_n$, a contradiction. The statements for meager, comeager and Baire property directly follow from the one for nowhere dense.

(2) Assume $A \times B$ is meager. By (1), there is a comeager $C \subseteq X$ such that for $x \in X$, the section $(A \times B)_x$ is meager in Y. If $A \subseteq C^c$ then A is meager, otherwise A meets C at a point x, and then the section $(A \times B)_x$ is B, so B is comeager.

(3) Assume that $H \subseteq X \times Y$ coincides with the open U outside the meager M and that H_x is meager for the generic x. Towards a contradiction, assume H is non-meager, so U is non-meager. By second-countability of $X \times Y$, there exist open $A \subseteq X$ and $B \subseteq Y$ such that $A \times B \subseteq H$ and $A \times B$ is not meager. By (2), A and B are non-meager. We assumed that H_x is meager for the generic $x \in X$, and by (1) M_x is meager for the generic $x \in X$. As A is non-meager, there exists $x \in A$ such that H_x and M_x are meager. As $B \subseteq H_x \cup M_x$, this contradicts B being non-meager.

□

(17.2) As an application of the Kuratowski–Ulam theorem, there exists a function $f : \mathbb{R} \to \mathbb{R}$ whose graph is a non-meager subset of $\mathbb{R}^2$.

Proof. Let $\left(M_\alpha, \alpha < 2^{\aleph_0}\right)$ be a transfinite enumeration of the meager F_σ subsets of $\mathbb{R}^2$. We construct f by transfinite stages indexed by ordinals $\alpha < 2^{\aleph_0}$. At stage α, we would like to fix the image of a new point to ensure that the graph of f is not contained in M_α. By the Kuratowski–Ulam theorem, there is a comeager subset $G \subseteq \mathbb{R}$ of reals x such that the section $(M_\alpha)_x := \{y \mid (x, y) \in M_\alpha\} \subseteq \mathbb{R}$ is meager. Being a comeager subset of the inhabited perfect Polish space $\mathbb{R}$, this G has cardinality continuum (¶ 15.7), and since we have fixed the values of f at less than continuum points in the previous steps, there is a point x such that $f(x)$ is still unset and $(M_\alpha)_x$ is meager. We set $f(x)$ to some real outside $(M_\alpha)_x$ (whose existence is guaranteed by the Baire category theorem). □

18. [Baire function classes]

(18.1) Let X be Polish. We define the class of ***Baire-α functions*** $f : X \to \mathbb{R}$: a Baire-0 function is a continuous function, and for $\alpha > 0$, a Baire-α function is a pointwise limit of functions each of which is Baire-β for some $\beta < \alpha$ (depending on the function). The ***Borel-α functions*** $f : X \to \mathbb{R}$ are those for which the preimage of an open subset is $\mathbf{\Sigma}^0_\alpha$. The main theorem of this section is that for all α, Baire-α functions coincide with Borel-$(\alpha + 1)$ functions.

TODO: write the rest of this section, for now see Kechris p. 190.

19. [Characterization of Baire-1 functions]

(19.1) Let X be Polish. We already know that Baire-1 functions $X \to \mathbb{R}$ coincide with Borel-2 functions. We prove another characterization: they are exactly those $f : X \to \mathbb{R}$ such that for every $F \subseteq X$ inhabited closed (equivalently, inhabited perfect), the restriction $f_{|F} : F \to \mathbb{R}$ has a continuity point.

TODO: write the rest of this section, for now see Kechris p. 193.

20. Topology refinements

The idea of topology refinements is to reduce the complexity of subsets by changing the topology to a finer one, the magic being that one can keep it Polish. Our first application of this powerful technique is the continuum hypothesis for Borel subsets.

This section proves in various places that certain Polish refinements of Polish topologies preserve Borel subsets, but we will later use these results to prove that this is in fact automatic.

(20.1) *Lemma:* Let X be a Polish space with topology τ, and for each n, let τ_n be a Polish topology refining τ. The topology τ_∞ generated by all the τ_n is Polish. Moreover, if each τ_n has the same Borel subsets as τ, then so does τ_∞.

Proof. To prove that τ_∞ is Polish, we realize X equipped with τ_∞ as a closed (hence G_δ) subspace of the Polish space $X^{\mathbb{N}}$, where the topology on $X^{\mathbb{N}}$ is the product topology of the τ_n. For this, we define the injection $\varphi : X \to X^{\mathbb{N}}$ by $\varphi(x) := (x, x, ...)$. This map is continuous (equipping the domain with the topology τ_∞), since the preimage of a basic open $U_0 \times \cdots \times U_{n-1} \times X \times X \times \cdots$ is $U_0 \cap \cdots \cap U_{n-1}$, which is open for τ_∞ as each U_i is open for τ_i and hence for τ_∞. Conversely, a basic open for τ_∞ is $U_0 \cap \cdots \cap U_{n-1}$ where each U_i is open for τ_i, and its image by φ is $(U_0 \times \cdots \times U_{n-1} \times X \times X \times \cdots) \cap \varphi(X)$, which is open in $\varphi(X)$; this shows that φ is a homeomorphism onto its image. Lastly, the image of φ is closed because if $(x_i \in X, i \in \mathbb{N})$ is such that $(\varphi(x_i), i \in \mathbb{N})$ converges to some $\ell \in \prod_n X$, i.e., (x_i) converges in each τ_n to ℓ_n, then ℓ_n is in particular a limit of (x_i) for τ, but τ is Hausdorff, so such a limit is unique, hence all ℓ_n are equal.

Now additionally assume that each τ_n has the same Borel subsets as τ. To prove that τ_∞ also has the same Borel subsets as τ, it suffices to prove that every subset which is open for τ_∞ is Borel for τ. The open subsets of τ_∞ are generated from open subsets of the τ_n, which are Borel for τ, by finite intersections and arbitrary unions, but these arbitrary unions can be replaced by countable unions by hereditary Lindelöfness of τ_∞. $\square$

(20.2) Let X be a Polish space with topology τ, and let the $B_n \subseteq X$ be countably many Borel subsets (for τ). There exists a Polish topology τ' on X, refining τ and having the same Borel subsets as τ, such that each B_n is clopen for τ'.

Proof. By the previous lemma, it suffices to prove this for just one Borel subset B.

We first prove the statement for B open. For this, we define τ' as the topology generated by τ and the extra closed subset B. This τ' is Polish because X with τ' is homeomorphic to the disjoint union of B and B^c with the sum topology, and both B and B^c are G_δ so Polish.

Next, we prove the statement for B Borel, by checking that subsets with this property are stable under complementation and countable unions. The case of complementation is trivial. Assume $B = \bigcup_n B_n$ where each B_n is Borel and let τ_n is a Polish refinement of τ preserving Borel subsets and making B_n clopen. By the lemma, the topology τ_∞ generated by the τ_n is Polish and also has the same Borel subsets as τ. Moreover, B is open for τ_∞. We finish off by applying the case of open subsets again to τ_∞ in order to make B closed as well. $\square$

(20.3) We deduce the **continuum hypothesis for Borel subsets**, also known as the **Alexandroff–Hausdorff theorem**: the cardinality of a Borel subset of a Polish space is countable or continuum.

Proof. With topology refinements, this is easy as pie: after refining the topology, it suffices to prove this for a clopen subset, which is Polish with the induced topology, and we already proved the continuum hypothesis for Polish spaces (¶ 11.3). □

(20.4) We can be a bit more precise: Let X be a Polish space and let $B \subseteq X$ be Borel uncountable. Then B embeds the Cantor space.

Proof. Call τ be the topology on X and let τ' be a Polish refinement of τ making B clopen. Then B with the topology induced by τ' is Polish, so it embeds the Cantor space through an injection $\varphi : 2^{\mathbb{N}} \to B$. The same injection $\varphi : 2^{\mathbb{N}} \to B$, with the target now topologized by τ, is still continuous because τ' refines τ, so it is a homeomorphism onto its image because the domain is compact and the target Hausdorff. □

(20.5) We can also make Borel functions continuous: if $f : X \to Y$ is a Borel function between Polish spaces, there exists a Polish refinement of the topology on X making f continuous.

Proof. Let τ be the topology on X. Take a countable basis (B_n) of Y, and let τ' be a Polish refinement of τ making each $f^{-1}(B_n)$ open. □

21. Analytic subsets

Lebesgue famously made a rookie mistake in his “proof” that projections of Borel subsets are Borel: he inadvertently used the completely wrong “fact” that projection commutes with intersection. Luzin and Suslin later found a counterexample, and this led to the theory of analytic subsets.

(21.1) A subset of a Polish space is ***analytic*** when it is the continuous image of some other Polish space, and ***co-analytic*** when its complement is analytic. The pointclasses of analytic and co-analytic subsets are respectively denoted $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Whether a subset is analytic only depends on the subspace topology it inherits. The spaces which are the continuous image of some Polish space are called Suslin spaces, so a subset is analytic iff it is a Suslin space with the induced topology. We will not make use of this remark, however.

(21.2) A Borel subset of a Polish space is analytic.

Proof. Let X be Polish with topology τ and let $B \subseteq X$ be Borel. We can find a Polish refinement τ' of τ making B clopen (¶ 20.2). Then B endowed with the topology induced by τ' is Polish and $\mathrm{id} : B \to X$ is continuous since τ' refines τ. □

(21.3) In practice, the usual way to prove that a subset A of a Polish X is analytic is to write A as the projection of a Borel subset of $X \times Y_0 \times \cdots \times Y_{n-1}$ for some other Polish $Y_0, \ldots, Y_{n-1}$. This is valid because Borel subsets are analytic and analytic subsets are trivially closed under continuous images. Informally, we find some equivalence

$$x \in A \Leftrightarrow \exists y_0 \in Y_0, \ldots, \exists y_{n-1} \in Y_{n-1}, \varphi(x, y_0, \ldots, y_{n-1})$$

where φ is a Borel condition. For a co-analytic subset, we need universal quantifiers instead.

The notation $\mathbf{\Sigma}^1_1(X)$ for the class of analytic subsets of X stems from the analogy with second-order arithmetic in logic: $\exists y \in Y$ is a sort of second-order quantifier (quantifying over the Cantor space $\{0,1\}^{\mathbb{N}}$ is equivalent to quantifying over subsets of $\mathbb{N}$) while quantifiers over countable sets are like first-order quantifiers. This connection is explored more precisely in effective descriptive set theory.

(21.4) *Examples.*

- For X Polish and $B \subseteq X$ Borel, the set of $K \in \mathcal{K}(X)$ which intersect B is analytic. Indeed, the condition is $\exists x \in X, x \in K \wedge x \in B$ and B is Borel while $\{(x, K) \in X \times \mathcal{K}(X) \mid x \in K\}$ is closed.
- The set

$$A := \{a \in \mathbb{N}^{\mathbb{N}} \mid \exists n_0 < n_1 < ..., \forall k \in \mathbb{N}, a(n_k) \mid a(n_{k+1})\}$$

  is analytic, because

$$a \in A \Leftrightarrow \exists s \in (\mathbb{N}^*)^{\mathbb{N}}, \forall k \in \mathbb{N}, a\left(\sum_{i=0}^{k} s(i)\right) \mid a\left(\sum_{i=0}^{k+1} s(i)\right).$$

  This is an existential quantifier on a Polish space followed by a countable quantifier, so we just need to check that for fixed $k \in \mathbb{N}$,

$$\left\{(a, s) \in \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^*)^{\mathbb{N}} \mid a\left(\sum_{i=0}^{k} s(i)\right) \mid a\left(\sum_{i=0}^{k+1} s(i)\right)\right\}$$

  is Borel, and indeed it is open.
- The set of differentiable functions in $\mathcal{C}([0,1])$ is co-analytic (and we will prove in § 30 that it is not Borel). Indeed, we claim that

$$\{(f, x) \in \mathcal{C}([0,1]) \mid f \text{ is differentiable at } x\}$$

  is Borel. To see this, we write the condition that the limit of $t_x(y) := (f(y) - f(x))/(y - x)$ as $y \to x$ exists as the equivalent of Cauchy sequences for functions:

$$f \text{ is differentiable at } x \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall yz \in [0,1],$$
$$|y - x| < \delta \wedge |z - x| < \delta \Rightarrow |t_x(z) - t_x(y)| < \varepsilon.$$

  The quantifiers over ε and δ can always be made to range over rationals without changing the truth of the statement, and because t_x is continuous as f is, the quantifiers on y and z also can.

(21.5) Analytic subsets of Polish spaces are closed under countable products, countable unions, countable intersections, Borel images and Borel preimages.

Proof. Countable products: If each X_n is Polish and each $A_n \subseteq X_n$ is analytic, say $A_n = f_n(Y_n)$ with Y_n Polish and f continuous, then $\prod_n A_n$ is the image of the induced $f : \prod_n Y_n \to \prod_n X_n$ (i.e., $f(y)_n := f_n(y_n)$).

Countable unions: If X is Polish and each $A_n \subseteq X$ is analytic, say $A_n = f_n(Y_n)$ with Y_n Polish and f_n continuous, then $\bigcup_n A_n$ is the image of the induced continuous function from the sum (disjoint union), $f : \left(\sum_n Y_n\right) \to X$.

Countable intersections: With the same notations, define

$$F := \left\{(y_n) \in \prod_n Y_n \mid \forall m, \forall n, f_m(y_m) = f_n(y_n)\right\}.$$

This is a closed subspace of $\prod_n Y_n$, hence Polish, and $\bigcap_n A_n$ is the image of the continuous function $F \to X$ mapping $(y_n) \in F$ to the common value of $f_n(y_n)$ for $n \in \mathbb{N}$.

Borel images: If Y and Z are Polish, $g : Y \to Z$ is Borel, and $A \subseteq X$ is analytic, say $A = f(X)$ where X is Polish and f is continuous, then we can refine the topology of X to a new Polish topology to make the Borel function $g \circ f : X \to Z$ continuous (¶ 20.5), and $g(A)$ is the image of this continuous function.

Borel preimages: If X and Z are Polish, $g : X \to Z$ is Borel, and $A \subseteq Z$ is analytic, say $A = f(Y)$ where Y is Polish and f is continuous, then $f^{-1}(A)$ is the image by the first projection (which is continuous) of $\{(x, y) \in X \times Y \mid g(x) = f(y)\}$, which is a Borel and hence analytic subset of $X \times Y$ (since it is the preimage of the diagonal $\{(z, z), z \in Z\}$, which is closed because Z is Polish hence Hausdorff, by the Borel function $X \times Y \to Z \times Z, (x, y) \mapsto (g(x), f(y)))$. □

(21.6) Next, some characterizations of analytic subsets. For a subset A of a Polish space X, the following are equivalent:

(1) A is analytic.
(2) A is empty or a continuous image of the Baire space.
(3) A is the projection of a closed subset of $X \times \mathbb{N}^{\mathbb{N}}$. (This is the characterization we will use the most often in the sequel.)
(4) A is the projection of a G_δ subset of $X \times 2^{\mathbb{N}}$.
(5) A is the projection of a G_δ subset of $X \times X$.

Proof. (1) $\Rightarrow$ (2) is because every inhabited Polish space is a continuous image of the Baire space (¶ 9.4). For (2) $\Rightarrow$ (3), the image of a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ is the projection of the graph of f with the coordinates swapped, which is a closed subset of $X \times \mathbb{N}^{\mathbb{N}}$. For (3) $\Rightarrow$ (4), the Baire space embeds into the Cantor space as a G_δ subset (¶ 6.2), and a closed subset of a G_δ subset is G_δ in a Polish space. For (4) $\Rightarrow$ (5), we distinguish two cases. If X is countable, then since singletons are closed in Hausdorff spaces, every subset of $X \times X$ is F_σ and dually G_δ. If X is uncountable, then it embeds the Cantor space (¶ 11.2), necessarily as a closed subset since the Cantor space is compact, and G_δ subsets of closed subsets are G_δ in Polish spaces. Finally, (6) $\Rightarrow$ (1) follows from the fact that Borel subsets are analytic (¶ 21.2) and analytic subsets are closed under continuous images (¶ 21.5). □

(21.7) Beyond analytic subsets, we define the ***projective hierarchy***. We already defined $\boldsymbol{\Sigma}^1_1(X)$ as the class of analytic subsets of X and $\boldsymbol{\Pi}^1_1(X)$ as the dual class of co-analytic subsets. For $n \geq 1$, we let $\boldsymbol{\Sigma}^1_{n+1}(X)$ contain the subsets which are the continuous image of a $\boldsymbol{\Pi}^1_n(Y)$ subset for some Polish Y, and we define $\boldsymbol{\Pi}^1_{n+1}$ subsets to be the complements of $\boldsymbol{\Sigma}^1_{n+1}$ subsets. Naturally, we define a subset to be $\boldsymbol{\Delta}^1_n$ when it is both $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$.

This extends the Borel hierarchy, of length ω_1, with a new hierarchy of length ω. Remarkably, it turns out that the $\boldsymbol{\Delta}^1_1$ subsets are actually just the Borel subsets, as we will show in ¶ 23.2.

The big picture is that Borel subsets are relatively well-understood, analytic and co-analytic subsets are somewhat well-understood, but already starting at $\boldsymbol{\Delta}^1_2$, many statements become independent of standard foundations and descriptive set theory gets more set-theoretic, exploring which assumptions such as large cardinals change the picture. For example, analytic subsets of $\mathbb{R}$ are Lebesgue-measurable, but it is independent (of, e.g., ZFC) whether all $\boldsymbol{\Delta}^1_2$ subsets of $\mathbb{R}$ are Lebesgue-measurable.

22. Universal analytic subsets

In § 13, we constructed universal $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ subsets of $X \times Y$ for X and Y Polish, Y uncountable, and deduced that the Borel hierarchy is strict through a diagonal argument. The story repeats: we construct universal analytic and co-analytic subsets, and deduce that not all analytic subsets are Borel.

(22.1) Let X be Polish and Y uncountable Polish. There exists a universal analytic subset of $X \times Y$ (and hence its complement is a universal co-analytic subset of $X \times Y$).

Proof. It suffices to find a universal analytic subset U of $X \times \mathbb{N}^\mathbb{N}$. With this in hand, we can embed the Baire space into Y (¶ 11.2), and this embedding makes the same U a subset of $X \times Y$ which is universal analytic (it remains analytic by reusing the same continuous function from a Polish space).

We can find a universal closed subset F of $(X \times \mathbb{N}^\mathbb{N}) \times \mathbb{N}^\mathbb{N}$ (¶ 13.3). Let A be the analytic subset of $X \times \mathbb{N}^\mathbb{N}$ obtained as the projection of F erasing the first $\mathbb{N}^\mathbb{N}$ component, i.e.,

$$A := \{(x, v) \in X \times \mathbb{N}^\mathbb{N} \mid \exists u \in \mathbb{N}^\mathbb{N}, ((x, u), v) \in F\}.$$

We claim that A is a universal analytic subset of $X \times \mathbb{N}^\mathbb{N}$. Indeed, an analytic subset $B \subseteq X$ can be written as the projection of some closed subset of $X \times \mathbb{N}^\mathbb{N}$ (¶ 21.6), which by universality of F can be written as the section $F^v := \{(x, u) \mid ((x, u), v) \in F\}$ for some $v \in \mathbb{N}^\mathbb{N}$, and then

$$\begin{aligned} B &= \{x \in X \mid \exists u \in \mathbb{N}^\mathbb{N}, (x, u) \in F^v\} \\ &= \{x \in X \mid \exists u \in \mathbb{N}^\mathbb{N}, ((x, u), v) \in U\} \\ &= \{x \in X \mid (x, v) \in A\}. \end{aligned}$$

□

(22.2) An uncountable Polish space X has an analytic subset which is not co-analytic, and in particular not Borel.

Proof. As for the strictness of the Borel hierarchy (¶ 13.4), we run a diagonal argument. We have a universal analytic subset A of $X \times X$ (¶ 22.1). Consider the subset $\{x \in X \mid (x, x) \in A\}$, which is analytic because it is the preimage of A by the continuous function $x \mapsto (x, x)$ (analytic subsets are closed under continuous preimages, ¶ 21.5). If its complement was analytic, universality of A would give $y \in X$ such that $\{x \in X \mid (x, x) \notin A\} = \{x \in X \mid (x, y) \in A\}$, and in particular $(y, y) \notin A \Leftrightarrow (y, y) \in A$, a contradiction. □

23. Luzin's separation theorem and consequences

We turn to the proof promised in ¶ 21.7 of Suslin's theorem that if a subset is analytic and co-analytic, then it is actually Borel (we already have the converse, ¶ 21.2). In fact we have the stronger result, known as Luzin's separation theorem, that disjoint analytic subsets can be separated by Borel subsets. We use it to derive several consequences purely on Borel subsets, illustrating the usefulness of analytic subsets for studying Borel subsets: every Borel bijection is a Borel isomorphism; a function is Borel iff its graph is Borel; and Borel subsets are closed under continuous injective images.

(23.1) We first recall the *bar induction principle*: Let X be a set, and let P be a predicate on the set $X^\star$ of finite sequences in X. Assume that:

(1) If a finite sequence s satisfies P, then every extension of s satisfies P.
(2) If every extension of a finite sequence s by just one more term satisfies P, then s already satisfies P.
(3) Every infinite sequence has a finite prefix satisfying P.

Then every finite sequence satisfies P.

Proof. Assume not. By (1), the empty sequence does not satisfy P. By (2), we can pick an element x_0 such that the one-element sequence (x_0) does not satisfy P, then x_1 such that (x_0, x_1) does not satisfy P, etc. Then the infinite sequence $(x_0, x_1, ...)$ contradicts (3). □

We will use the following variant, a "double bar induction principle": Let X and Y be sets. Let P be a predicate on pairs of sequences $s \in X^\star, t \in Y^\star$ such that:

(1) For all finite sequences $s \in X^\star, t \in Y^\star$, if $P(s, t)$ holds, then $P(\hat{s}, \hat{t})$ holds for all extensions $\hat{s}$ of s and $\hat{t}$ of t.
(2) For all finite sequences $s \in X^\star, t \in Y^\star$, if $P(s \cdot (x), t)$ holds for all $x \in X$, then $P(s, t)$ holds.
(3) For all finite sequences $s \in X^\star, t \in Y^\star$, if $P(s, t \cdot (y))$ holds for all $y \in Y$, then $P(s, t)$ holds.
(4) For all infinite sequences $u \in X^{\mathbb{N}}, v \in Y^{\mathbb{N}}$, there exist a finite prefix s of u and a finite prefix t of v such that $P(s, t)$ holds.

Then $P(s, t)$ holds for all s, t.

Proof. Assume not. By (1), $P(\varepsilon, \varepsilon)$ does not hold. By (2), we can pick x_0 such that $P((x_0), \varepsilon)$ does not hold. By (3), we can pick y_0 such that $P((x_0), (y_0))$ does not hold. We continue repeatedly, building sequences $(x_n), (y_n)$. By (4), there exist m, n such that $P((x_0, ..., x_{m-1}), (y_0, ..., y_{n-1})$ does not hold. Let $p := \min(m, n)$. By (1), $P\big((x_0, ..., x_{p-1}), (y_0, ..., y_{p-1})\big)$ does not hold, a contradiction. □

(23.2) Luzin's separation theorem: Let X be a Polish space and let $A, A' \subseteq X$ be disjoint analytic subsets. There exists a Borel subset B which separates A and A', i.e., $A \subseteq B$ and $A' \subseteq B^c$.

Proof. We have continuous functions $f, f' : \mathbb{N}^{\mathbb{N}} \to X$ such that A and A' are respectively the image of f and f'. We apply the previous "double bar induction principle" where for $s, t \in \mathbb{N}^\star$, we define $P(s, t)$ as the property "there exists a Borel subset separating $f([s])$ and $f'([t])$". We verify the hypotheses:

(1) Trivial.
(2) If $f([s])$ is separated from each $f'([t \cdot (n)])$ by a Borel subset B_n, then $f([s])$ is separated from $f'([t]) = \bigcup_n f'([t \cdot (n)])$ by the Borel subset $\bigcap_n B_n$.
(3) If each $f([s \cdot (n)])$ is separated from $f'([t])$ by a Borel subset B_n, then $f([s]) = \bigcup_n f([s \cdot (n)])$ is separated from $f'([t])$ by the Borel subset $\bigcup_n B_n$.
(4) Let $u, v \in \mathbb{N}^{\mathbb{N}}$. We have $f(u) \neq f'(v)$ since $f(u) \in A$ and $f'(v) \in A'$, and A is disjoint from A'. As X is Hausdorff, we have disjoint neighborhoods U of $f(u)$ and V of $f'(v)$. By continuity of f, there is a finite prefix s of u such that $f([s]) \subseteq U$. Likewise, there is a finite prefix t of v such that $f'([t]) \subseteq V$. Then the open subset U (or V) separates $f([s])$ from $f'([t])$.

□

(23.3) Suslin's theorem: If a subset of a Polish space is analytic and co-analytic, then it is Borel (and conversely, ¶ 21.2).

Proof. Use Luzin's separation theorem (¶ 23.2) to separate the subset from its complement. □

(23.4) An example application of Suslin's theorem: A *two-point set* is a subset of the plane which meets every line in exactly two points. (Exercise: construct a two-point set using a transfinite enumeration of all lines in the plane.) It is unknown (as of August 2026) whether there exists a Borel two-point set, but if a two-point set is analytic, then it must be Borel.

Proof. Let $H \subseteq \mathbb{R}^2$ be an analytic two-point set. In fact we only need that it meets every *vertical* line in exactly two points. We have

$$(x, y) \in H^c \Leftrightarrow \exists z, t \in \mathbb{R}, (x, z) \in H \wedge (x, t) \in H \wedge z \neq t \wedge y \neq z \wedge y \neq t$$

This shows that H^c is analytic, so H is Borel by Suslin's theorem. □

(23.5) A Borel bijection between Polish spaces is a Borel isomorphism (its inverse is Borel as well).

Proof. Let $f : X \to Y$ be a Borel bijection between Polish spaces. For every Borel $B \subseteq X$, the image $f(B)$ is analytic, and since f is a bijection, $f(B)^c = f(B^c)$ so $f(B)$ is also co-analytic, hence Borel by Suslin's theorem (¶ 23.3). (This is also an immediate corollary of the graph theorem, ¶ 23.6, or of the Luzin–Suslin theorem, ¶ 23.7.) □

(23.6) Graph theorem: Let $f : X \to Y$ be a function between Polish spaces. The following are equivalent:

(1) f is a Borel function.
(2) The graph of f is a Borel subset of $X \times Y$.
(3) The graph of f is an analytic subset of $X \times Y$.

Proof. (1) ⟹ (2): Let (B_n) be a countable basis of Y. The graph of f is defined by a Borel condition:

$$\begin{aligned}(x, y) \in \mathrm{graph}(f) &\Leftrightarrow f(x) = y \\ &\Leftrightarrow \forall n, y \in V_n \Leftrightarrow f(x) \in V_n \\ &\Leftrightarrow \forall n, y \in V_n \Leftrightarrow x \in f^{-1}(V_n).\end{aligned}$$

(3) ⟹ (1): If $\mathrm{graph}(f) \subseteq X \times Y$ is analytic, then for all Borel $B \subseteq Y$, the preimage $f^{-1}(B)$ is analytic because it is the projection of $\mathrm{graph}(f) \cap (X \times B)$, and it is also co-analytic because its complement is the projection of $\mathrm{graph}(f) \cap (X \times B^c)$, so it is Borel by Suslin's theorem (¶ 23.3). □

(23.7) Luzin-Suslin theorem: Borel subsets of Polish spaces are closed under continuous injective images.

Proof. TODO; for now see Kechris p. 89 □

24. Analytic subsets through Suslin's operation $\mathcal{A}$

Before we move on from analytic subsets, we mention one more characterization. We used a topological definition of analytic subsets, but their whole theory can be developed more "algebraically" from this definition.

(24.1) Let X be a set and let $H = (H_s, s \in \mathbb{N}^\star)$ be a family of subsets of X, indexed by finite words over $\mathbb{N}$. We define

$$\mathcal{A}(H) := \bigcup_{s\in\mathbb{N}^\mathbb{N}} \bigcap_{n\in\mathbb{N}} H_{s[:n]}.$$

This is called ***Suslin's operation $\mathcal{A}$***. We picture the elements of $\mathbb{N}^\star$ as the nodes of an infinite countably branching tree, the Baire tree. The family H is a labeling of nodes by subsets of X, and $\mathcal{A}(H)$ is the set of points which are in the intersection of the labels of all the nodes in some infinite branch.

(24.2) Let X be Polish. A subset $A \subseteq X$ is analytic iff there exists a family of closed subsets $H = (H_s, s \in \mathbb{N}^\star)$ such that $A = \mathcal{A}(H)$.

Proof. Assume A is analytic, so A is the image of some continuous $f : \mathbb{N}^\mathbb{N} \to X$. We set $H_s := \mathrm{cl}(f([s]))$ and claim $A = \mathcal{A}(H)$. The direction $A \subseteq \mathcal{A}(H)$ is immediate. Let $x \in \mathcal{A}(H)$, so we have $s \in \mathbb{N}^\mathbb{N}$ such that $x \in \mathrm{cl}(f([s[:n]]))$ for all n. The continuity of f at s means that for all $\varepsilon > 0$, there exists n such that $f([s[:n]]) \subseteq B(f(s), \varepsilon)$, and in particular

$$x \in \mathrm{cl}(f([s[:n]])) \subseteq \mathrm{cl}(B(f(s), \varepsilon)) \subseteq \overline{B}(f(s), \varepsilon),$$

so $d(x, f(s)) \leq \varepsilon$ for all $\varepsilon > 0$, so $x = f(s)$ and thus $x \in A$.

Conversely, if each H_s is closed, $\mathcal{A}(H)$ is analytic because

$$x \in \mathcal{A}(H) \Leftrightarrow \exists s \in \mathbb{N}^\mathbb{N}, \forall n \in \mathbb{N}, x \in H_{s[:n]}$$

and the condition under $\exists$ is closed. □

(24.3) Let X be a set and let $\mathcal{H} \subseteq \mathcal{P}(X)$. Let $\mathcal{A}(\mathcal{H})$ denote the class of subsets of the form $\mathcal{A}(H)$ for a family $H := (H_s \in \mathcal{H}, s \in \mathbb{N}^\star)$. A bit of manipulation of trees shows that this operation is idempotent: $\mathcal{A}(\mathcal{A}(\mathcal{H})) = \mathcal{A}(\mathcal{H})$. Namely, $\mathcal{A}\big(\mathcal{A}\big(H^{(s)}\big), s \in \mathbb{N}^\star\big) = \mathcal{A}(H'_u, u \in \mathbb{N}^\star)$ where

$$\begin{aligned}
&H'_\varepsilon := H'^{(\varepsilon)}_\varepsilon \\
&H'_{u_0} := H'^{(\varepsilon)}_{u_0} \\
&H'_{u_0,u_1} := H'^{(u_1)}_\varepsilon \\
&H'_{u_0,u_1,u_2} := H'^{(\varepsilon)}_{u_0,u_2} \\
&H'_{u_0,u_1,u_2,u_3} := H'^{(u_1)}_{u_3} \\
&H'_{u_0,u_1,u_2,u_3,u_4} := H'^{(u_4)}_\varepsilon \\
&\vdots
\end{aligned}$$

(so an infinite branch $u_0, u_1, u_2, \ldots$ is essentially decoded as an infinite branch in the main tree interleaved with one infinite branch for each subtree encountered on the way).

In particular, subsets obtained from analytic subsets by the $\mathcal{A}$ operation are still analytic.

Another consequence is that an analytic subset can also obtained through the $\mathcal{A}$ operation applied to a family of open subsets, rather than closed subsets. Indeed, a closed subset F of a Polish space is G_δ, say $F = \bigcap_{n\in\mathbb{N}} U_n$ with U_n open, and thus can be expressed as $F = \mathcal{A}(H_s)$ where $H_s := U_{|s|}$ is open (we label all nodes on the same level n of the Baire tree by U_n).

25. Standard Borel spaces

We are already familiar with the idea that Polish spaces are just *metrizable* spaces and not *metric* spaces: it is the topology that matters and not the particular metric that generates it. Often, we want to go further and consider only the Borel structure, namely the σ-algebra of Borel subsets, and not the particular topology it is generated from. We prove here that, amazingly, the Borel structure is the same on *all* uncountable Polish spaces, which is a motivation for choosing Polish spaces as the "nice" class of topological spaces for descriptive set theory since they offer the flexibility to study the Borel structure of classical spaces like $\mathbb{R}$ and $\mathbb{N}^{\mathbb{N}}$ using more complicated spaces like $\mathcal{C}([0,1])$ and $\mathcal{K}([0,1])$ while remaining tame enough not to induce different Borel structures (except the degenerate Borel structure on a countable Polish space, where all subsets are Borel, indeed F_σ).

(25.1) All Polish spaces of the same cardinal are Borel-isomorphic.

Proof. For countable Polish spaces, this is clear (singletons are closed, so every subset is F_σ). It thus suffices to prove that every uncountable Polish space X is Borel-isomorphic to the Cantor space. We have an embedding $\varphi : \{0,1\}^{\mathbb{N}} \hookrightarrow X$, which is a (continuous hence) Borel injection (¶ 11.2). Conversely, we have an embedding $\psi : X \hookrightarrow \mathbb{R}^{\mathbb{N}}$ (¶ 9.1), and we can obtain a Borel injection $\mathbb{R}^{\mathbb{N}} \to (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$ by mapping every real to its proper binary expansion.

We conclude by a "Cantor–Schröder-Bernstein lemma for Borel functions": If X and Y are two Polish spaces such that there exist Borel injections $f : X \to Y$ and $g : Y \to X$, then X and Y are Borel-isomorphic.

For this, we reuse the standard proof of the Cantor–Schröder–Bernstein theorem, checking that the constructed map is Borel. We observe the iterations of the endofunction induced by f and g on the disjoint union $X + Y$. This partitions $X + Y$ into "chains" of four types:

- One-sided, starting in X:

$$x_0 \in X \overset{f}{\mapsto} y_0 \in Y \overset{g}{\mapsto} x_1 \in X \overset{f}{\mapsto} y_1 \overset{g}{\mapsto} \cdots$$

- One-sided, starting in Y:

$$y_0 \in Y \overset{g}{\mapsto} x_0 \in X \overset{f}{\mapsto} y_1 \in Y \overset{g}{\mapsto} x_1 \overset{f}{\mapsto} \cdots$$

- Double-sided:

$$\cdots \overset{g}{\mapsto} x_{-1} \in X \overset{f}{\mapsto} y_{-1} \in Y \overset{g}{\mapsto} x_0 \in X \overset{f}{\mapsto} y_0 \in Y \overset{g}{\mapsto} x_1 \in X \overset{f}{\mapsto} y_1 \in Y \overset{g}{\mapsto} \cdots$$

- Cycles:

$$x_0 \in X \overset{f}{\mapsto} y_0 \in Y \overset{f}{\mapsto} \cdots \overset{g}{\mapsto} x_{n-1} \in X \overset{f}{\mapsto} y_{n-1} \in Y \overset{g}{\mapsto} x_0$$

On double-sided chains and cycles, f and g both induce bijections between the elements from X and the elements from Y, while on one-sided chains starting in X (resp. Y), only f (resp. g) does. We can thus define a bijection $\varphi : X \to Y$ by using f on the subset B of elements from one-sided chains started in X, and g^{-1} on B^c (where it is indeed defined).

We have to check that φ is Borel (the verification that φ^{-1} is Borel would be similar, and in any case it is automatic since a Borel bijection between Polish spaces is a Borel isomorphism, ¶ 23.5). First, B is Borel because

$$B = \bigcup_{n \in \mathbb{N}} (g \circ f)^{-n}(g(Y)^c)$$

and $g(Y)$ is Borel by the Luzin–Suslin theorem (Borel subsets of Polish spaces are preserved by injective Borel images, ¶ 23.7). We deduce that when $H \subseteq Y$ is Borel,

$$\varphi^{-1}(H) = (B \cap f^{-1}(H)) \cap (B^c \cap g(H))$$

is Borel, using the Luzin–Suslin theorem again for $g(H)$. □

(25.2) A ***standard Borel space*** is a Polish space viewed up to Borel isomorphism. In other words, it is a set X together with a σ-algebra $\mathcal{S}$ on X (i.e., a measurable space, but we are not doing measure theory) such that there exists some Polish topology on X for which $\mathcal{S}$ is the set of Borel subsets of X.

By the continuum hypothesis for Polish spaces (¶ 11.3) and the fact that Polish spaces of the same cardinal are Borel-isomorphic (¶ 25.1), the only standard Borel spaces are, up to isomorphism, the uninteresting Borel spaces on a finite or countably infinite set where every subset is Borel, and the Borel space of any uncountable Polish space.

(25.3) TODO: analytic subsets in a standard Borel space

(25.4) An important example is the ***Effros Borel space*** $\mathcal{F}(X)$ of a Polish space X (possibly countable). It consists of the closed subsets of X (so $\mathcal{F}(X) = \mathbf{\Sigma}^0_1(X)$ as sets, but the notation $\mathcal{F}(X)$ emphasizes the Effros Borel space structure), and its σ-algebra is generated by the subsets $\{F \in \mathcal{F}(X) \mid F \text{ intersects } U\}$ for $U \subseteq X$ open. This is a standard Borel space.

Proof. TODO; for now see Kechris p. 75. □

26. [Gale–Stewart games]

TODO

27. [The Banach–Mazur game]

TODO

28. [Analytic determinacy holds under a measurable cardinal]

TODO

29. [Analytic determinacy fails in the constructible universe]

TODO

30. [Complete analytic and co-analytic subsets]

(30.1) TODO: subsets complete for a pointclass; Borel reductions

(30.2) TODO: ill-founded trees form an analytic-complete subset of trees

(30.3) TODO: differentiable functions form a co-analytic-complete subset of $\mathcal{C}([0,1])$

31. Uniformization theorems

(31.1) Let X, Y be sets and let $H \subseteq X \times Y$. A ***uniformization*** of H is a subset $U \subseteq H$ such that for every $x \in \mathrm{pr}_1(H)$ (where $\mathrm{pr}_1 : X \times Y \to X$ is the first projection), there exists a unique $y \in Y$ such that $(x, y) \in H$. A ***uniformizing function*** for H is a function $f : \mathrm{pr}_1(H) \to Y$ such that $(x, f(x)) \in H$ for all $x \in \mathrm{pr}_1(H)$. So a uniformization is just the graph of a uniformizing function, but it can be useful to view it as a subset when talking about its topological properties.

(31.2) The following question often comes up: do "nice" subsets have "nice" uniformizations? Unfortunately, if X and Y are Polish and $H \subseteq X \times Y$ is Borel, it does not follow that H has a Borel uniformization. Indeed, if $A \subseteq X$ is analytic but not Borel, there is a closed subset $F \subseteq A \times \mathbb{N}^{\mathbb{N}}$ such that $A = \mathrm{pr}_1(F)$, but if F had a Borel uniformization U, then A would be the continuous injective image of U by $(\mathrm{pr}_1)_{|U} : U \to A$, hence Borel.

(31.3) We now state four theorems giving conditions under which nice uniformizations exist. We will only prove the last one, but the others use the same idea.

(31.4) Small sections uniformization theorem: Let X, Y be Polish and let $H \subseteq X \times Y$ be Borel. If each section $H_x, x \in X$ is σ-compact (i.e., a countable union of compact subsets of Y), then H has a Borel uniformization.

(31.5) Large sections uniformization theorem: Let X, Y be Polish and let $H \subseteq X \times Y$ be Borel.

- If each section $H_x, x \in X$ is empty or non-meager, then H has a Borel uniformization.
- If there exists a finite measure on Y with its Borel σ-algebra such that each section $H_x, x \in X$ is empty or has positive measure, then H has a Borel uniformization.

(31.6) Co-analytic uniformization theorem: Let X, Y be Polish and let $H \subseteq X \times Y$ be co-analytic. Then H has a co-analytic uniformization. (We say that "$\mathbf{\Pi}^1_1$ has the uniformization property".)

(31.7) Jankov–von Neumann uniformization theorem: Let X, Y be Polish and let $H \subseteq X \times Y$ be analytic. Then H has a uniformization U which lies in the σ-algebra generated by the analytic subsets of $X \times Y$, written $U \in \sigma(\mathbf{\Sigma}^1_1)$.

Proof. In several steps:

(1) Let X be Polish and let $F \subseteq X \times \mathbb{N}^{\mathbb{N}}$ be closed. We claim that F has a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function, i.e., there exists $f : \mathrm{pr}_1(F) \to \mathbb{N}^{\mathbb{N}}$ such that $(x, f(x)) \in F$ for all x and preimages of opens by f are $\sigma(\mathbf{\Sigma}^1_1)$ in X.

For this, we let f map $x \in \mathrm{pr}_1(F)$ to the ***leftmost branch*** of the section F_x, which is a closed inhabited subset of $\mathbb{N}^{\mathbb{N}}$. This is the sequence whose first term u_0 is the minimum first term among all sequences in F, whose second term u_1 is the minimum second term among all sequences in F having first term u_0, etc. It is in F_x because F_x is closed, which shows that f is a uniformizing function. To see that f is $\sigma(\mathbf{\Sigma}^1_1)$-measurable, it suffices to check that for every $s \in \mathbb{N}^\star$, the preimage of the basic open $[s]$ (the set of sequences having s as a prefix) is $\sigma(\mathbf{\Sigma}^1_1)$ (then we write an open as a countable union of basic opens using the hereditary Lindelöfness of Y). This holds because, writing $s = s_0 ... s_{k-1}$, we have

$$f^{-1}([s]) = \mathrm{pr}_1(F \cap (X \times [s])) \cap \bigcap_{\ell=0}^{k-1} \bigcap_{r=0}^{s_\ell - 1} \mathrm{pr}_1(F \cap (X \times [s_0 ... s_{\ell-1} r]))^c$$

(the leftmost branch of a tree starts with s iff the tree has a branch starting with s and no branch starting with a lexicographically smaller sequence). This is $\sigma(\mathbf{\Sigma}^1_1)$ because each $\mathrm{pr}_1(F \cap (X \times [...]))$ is the projection of a closed subset, hence analytic.

(2) Let X, Y be Polish and let $F \subseteq X \times Y$ be closed. We claim that F has a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function.

For this, write $Y = \varphi(\mathbb{N}^{\mathbb{N}})$ where $\varphi : \mathbb{N}^{\mathbb{N}} \to Y$ is continuous surjective (we know that every Polish space is a continuous image of the Baire space), and let

$$F' := \{(x, u) \mid (x, \varphi(u)) \in F\} \subseteq X \times \mathbb{N}^{\mathbb{N}}.$$

This is closed, so the previous point gives a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function $f' : \mathrm{pr}_1(F') \to \mathbb{N}^{\mathbb{N}}$ of F. Then $\mathrm{pr}_1(F') = \mathrm{pr}_1(F)$ because φ is surjective, and $\varphi \circ f'$ is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function for F.

(3) Let X, Y be Polish and let $A \subseteq X \times Y$ be analytic. We claim that A has a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function.

For this, write $A = \mathrm{pr}_{1,2}(F)$ where $F \subseteq X \times Y \times \mathbb{N}^{\mathbb{N}}$ is closed. The previous theorem gives us a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function $f : \mathrm{pr}_1(F) \to Y \times \mathbb{N}^{\mathbb{N}}$ for F. Then $\mathrm{pr}_1 \circ f$ is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function for A.

(4) It remains to check that we can go from a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function to a $\sigma(\mathbf{\Sigma}^1_1)$ uniformizing subset. For this, we prove that if X and Y are Polish, $A \subseteq X$ is analytic, and $f : A \to Y$ is $\sigma(\mathbf{\Sigma}^1_1)$-measurable, then $\mathrm{graph}(f)$ is $\sigma(\mathbf{\Sigma}^1_1)$ in $X \times Y$. This is done by taking a countable basis (U_n) of Y and observing that

$$\begin{aligned}(x, y) \in \mathrm{graph}(f) &\Leftrightarrow (x \in A \wedge f(x) = y) \\ &\Leftrightarrow (x \in A \wedge \forall n \in \mathbb{N}, f(x) \in U_n \Leftrightarrow y \in U_n)\end{aligned}$$

which is a $\sigma(\mathbf{\Sigma}^1_1)$ condition.

$\square$